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Gale-Nikaido lemma and Sperner's lemma: A constructive analysis[1]

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Abstract

We will show a constructive (or an approximate) version of the Gale-Nikaido lemma, which is the basis for the proof of the existence of an approximate equilibrium in a competitive economy, by Sperner's lemma. We also show that a constructive version of the Gale-Nikaido lemma leads to Sperner's lemma. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

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Keywords: Gale-Nikaido lemma, competitive equilibrium, Sperner's lemma, constructive mathematics.

1. Introduction

It is often said that Brouwer's fixed point theorem can not be constructively proved.

> [6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics a la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [2] or [4]. Brouwer's fixed point theorem can be constructively, in the sense of constructive mathematics a la Bishop, proved only approximately. The existence of an exact fixed point of a function which satisfies some property of local non-constancy may be constructively proved.

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Thus, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) with closed graph and the existence of an equilibrium in a competitive economy with multi-valued demand and supply functions also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. See [4] and [9]. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. It seems that we can also constructively prove a constructive version of Kakutani's fixed point theorem, and using this theorem we can prove the existence of an approximate equilibrium in a competitive economy with multi-valued demand and supply functions.

> Then, can we prove the existence of an approximate equilibrium in a competitive economy directly by Sperner's lemma?

We present such a proof. We will show an approximate version of the Gale-Nikaido lemma ([5] and [7]), which is the basis for the proof of the existence of an approximate equilibrium in a competitive economy with multi-valued demand and supply functions, by Sperner's lemma, and we also show that an approximate version of the Gale-Nikaido lemma leads to Sperner's lemma.

The Gale-Nikaido lemma states the following result. Let Δ^n be an n-dimensional simplex and Z be a totally bounded and complete, that is, compact and convex set in $n+1$-dimensional Euclidian space with inhabited (nonempty) interior. Suppose that a multi-function F from Δ^n to the set of inhabited subsets of Z satisfies some conditions including the Weak Walras Law and the property of closed graph. Then, for some $\mathbf{p}^* \in \Delta^n$ there exists $\mathbf{z}^* \in Z$ which satisfies

$$\mathbf{z}^* \in F(\mathbf{p}^*), \text{ and } \mathbf{z}^* \leq 0.$$

We will show that under similar conditions the following result holds.

> Let $\varepsilon > 0$. For some $\mathbf{p}^* \in \Delta^n$ there exists $\mathbf{z}^*$ which satisfies
>
> $$|F(\mathbf{p}^*) - \mathbf{z}^*| < \varepsilon, \text{ and } \mathbf{z}^* < \varepsilon\mathbf{e}.$$

$\mathbf{e}$ is a vector whose each component is 1. About the closed graph property of multi-functions we consider its uniform version, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph.

In the next section we prove Sperner's lemma. This proof is a standard constructive proof. In Section 3 we prove an approximate version of the Gale-Nikaido lemma by Sperner's lemma. In Section 4 we will show that an approximate version of the Gale-Nikaido lemma leads to Sperner's lemma. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

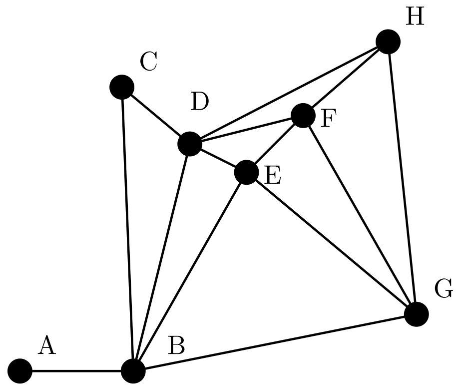

Figure 1: Example of graph

2. Sperner's lemma

To prove Sperner's lemma we use the following simple result in graph theory, Handshaking lemma. A *graph* refers to a collection of vertices and a collection of edges that connect pairs of vertices. Each graph may be undirected or directed. Figure 1 is an example of an undirected graph. The degree of a vertex of a graph is defined to be the number of edges incident to the vertex, with loops counted twice. Each vertex has odd degree or even degree. Let v denote a vertex and V denote the set of all vertices.

Lemma 2.1. [Handshaking lemma] Every undirected graph contains an even number of vertices of odd degree. That is, the number of vertices that have an odd number of incident edges must be even.

It is a simple lemma. But for completeness of arguments we provide a proof.

Proof. Prove this lemma by double counting. Let $d(v)$ be the degree of vertex v. The number of vertex-edge incidences in the graph may be counted in two different ways: by summing the degrees of the vertices, or by counting two incidences for every edge. Therefore

$$\sum_{v \in V} d(v) = 2e,$$

where e is the number of edges in the graph. The sum of the degrees of the vertices is therefore an even number. It could happen if and only if an even number of the vertices had odd degree. ■

Let Δ^n denote an n-dimensional simplex. n is a positive integer at least 2. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Figure 2 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ^n in m equal segments, and draw the lines parallel to the sides of Δ^n. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ^n inductively for cases of higher dimension. In a 3 dimensional case each face of Δ^n is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ^n. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ^n constructed by partition. The vertices of these small simplices of K are labeled with the numbers 0, 1, 2, …, n subject to the following rules.

1. The vertices of Δ^n are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, …, a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.

2. If a vertex of simplices of K is contained in an $n-1$-dimensional face of Δ^n, then that vertex is labeled with some number which is the same as the number of a vertex of that face.

3. If a vertex of simplices of K is contained in an $n-2$-dimensional face of Δ^n, then that vertex is labeled with some number which is the same as the number of a vertex of that face. And similarly for cases of lower dimension.

4. A vertex contained in inside of Δ^n is labeled with arbitrary number among 0, 1, …, n.

A small simplex of K which is labeled with the numbers 0, 1, …, n is called a *fully labeled simplex*. Now let us prove Sperner's lemma.

Lemma 2.2. [Sperner's lemma] If we label the vertices of simplices of K following above rules (1) $\sim$ (4), then there are an odd number of fully labeled simplices. Thus, there exists at least one fully labeled simplex.
Proof. See Appendix 5. ∎

Since n and partition of Δ^n are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n-dimensional simplex of K through finite steps.

3. Approximate version of the Gale-Nikaido lemma

In this section we derive an approximate version of the Gale-Nikaido lemma which is the basis for the proof of the existence of an approximate equilibrium in a competitive economy with multi-valued demand and supply functions. The contents of (the classical version of) the Gale-Nikaido lemma are as follows.

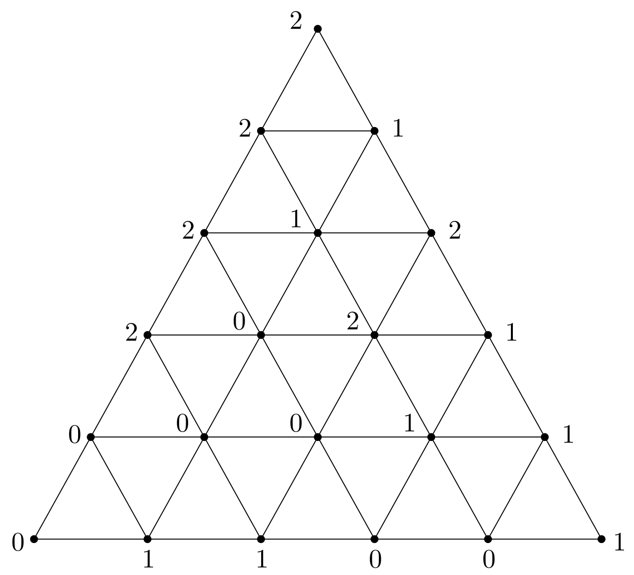

Figure 2: Partition and labeling of 2-dimensional simplex

Gale-Nikaido lemma

Let $\mathbf{p} = (p_0, p_1, \ldots, p_n)$ and

$$\Delta^n = \left\{ \mathbf{p} | p_i \geq 0,\ i = 0, 1, \ldots, n,\ \sum_{i=0}^{n} p_i = 1 \right\},$$

and let Z be a compact and convex set with inhabited interior in $n + 1$-dimensional Euclidian space. Assume that a multi-function $F(\mathbf{p})$ from Δ^n to the set of inhabited subsets of Z satisfies the following conditions.

1. $F(\mathbf{p})$ is a compact and convex set of Z for each $\mathbf{p}$.

2. F has a closed graph.

3. (Weak Walras Law) For any $\mathbf{p} \in \Delta^n$ and $\mathbf{z} \in Z$, $\mathbf{pz} \leq 0$ holds.

Then, for some $\mathbf{p}^* \in \Delta^n$ there exists $\mathbf{z}^*$ which satisfies

$$\mathbf{z}^* \in F(\mathbf{p}^*),\ \mathbf{z}^* \leq 0.$$

A graph of a multi-function F from Δ^n to the set of inhabited subsets of Z is

$$G(F) = \cup_{\mathbf{p} \in \Delta^n} \{\mathbf{p}\} \times F(\mathbf{p}).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact.

> Consider sequences $(\mathbf{p}_n)_{n\geq 1}$ and $(\mathbf{q}_n)_{n\geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$. If $\mathbf{p}_n \longrightarrow \mathbf{p}$ and $\mathbf{q}_n \longrightarrow \mathbf{q}$, then $\mathbf{q} \in F(\mathbf{p})$.

According to [3] this means

> If for each neighborhood $U(\mathbf{p}, \varepsilon)$ of $\mathbf{p}$ there exists n_0 such that $\mathbf{p}_n \in U(\mathbf{p}, \varepsilon)$ when $n \geq n_0$, then for the union of neighborhoods $\cup_{\mathbf{q}\in F(\mathbf{p})} V(\mathbf{q}, \varepsilon)$ of points in $F(\mathbf{p})$ there exists n_0' such that $\mathbf{q}_n \in \cup_{\mathbf{q}\in F(\mathbf{p})} V(\mathbf{q}, \varepsilon)$ when $n \geq n_0'$.

Further we consider a uniform version of this property for multi-functions, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. It means that n_0 and n_0' depend on only ε not on $\mathbf{p}$.

Let $\mathbf{z} = (z_0, z_1, \dots, z_n)$, and consider the following function.

$$\varphi(\mathbf{p}, \mathbf{z}) = (\varphi_0, \varphi_1, \dots, \varphi_n),\ \varphi_i(\mathbf{p}, \mathbf{z}) = \frac{p_i + \max(z_i, 0)}{1 + \sum_{j=0}^{n} \max(z_j, 0)}.$$

Since we have $\varphi_i \geq 0$, $\sum_{i=0}^{n} \varphi_i = 1$, and φ_i is a uniformly continuous function of $(\mathbf{p}, \mathbf{z})$, $\varphi(\mathbf{p}, \mathbf{z})$ is a uniformly continuous function from $\Delta^n \times Z$ to Δ^n. And because $F(\mathbf{p})$ is convex, $\varphi(\mathbf{p}, \mathbf{z}) \times F(\mathbf{p})$ is also a convex set of $\Delta^n \times Z$.

Next we define the following multi-function,

$$g(\mathbf{p}, \mathbf{z}) = \varphi(\mathbf{p}, \mathbf{z}) \times F(\mathbf{p}). \tag{3.1}$$

$g(\mathbf{p}, \mathbf{z})$ is a multi-function from $\Delta^n \times Z$ to the set of inhabited subsets of $\Delta^n \times Z$. $\varphi(\mathbf{p}, \mathbf{z})$ is a single-valued function, and it is a special case of multi-function. Δ^n itself is a subset of the set of inhabited subsets of Δ^n, and so we can consider that the set of inhabited subsets of Δ^n is the range of $\varphi(\mathbf{p}, \mathbf{z})$. Thus, g is considered to be a multi-function from $\Delta^n \times Z$ to the set of inhabited subsets of $\Delta^n \times Z$. Since $\varphi(\mathbf{p}, \mathbf{z})$ is a uniformly continuous function, and $F(\mathbf{p})$ is a multi-function with uniformly closed graph, $g(\mathbf{p}, \mathbf{z})$ also uniformly has a closed graph. Since Z is homeomorphic to an $n + 1$-dimensional simplex, $\Delta^n \times Z$ is homeomorphic to an $2n + 1$-dimensional simplex.

In contrast to the classical version of the Gale-Nikaido lemma, we call the following result an *approximate version* of the Gale-Nikaido lemma.

Theorem 3.1. [Approximate version of the Gale-Nikaido lemma] Assume the same conditions, 1, 2 and 3, replacing 2 with that F uniformly has a closed graph. Let $\varepsilon > 0$. For some $\mathbf{p}^* \in \Delta^n$ there exists $\mathbf{z}^*$ which satisfies

$$|F(\mathbf{p}^*) - \mathbf{z}^*| < \varepsilon, \text{ and } \mathbf{z}^* < \varepsilon\mathbf{e}.$$

$\mathbf{e}$ is a vector whose each component is 1. $\mathbf{p}^*$ and $\mathbf{z}^*$ depend on ε.

Proof. Let consider a multi-function with uniformly closed graph Θ from Δ^{2n+1} to the set of its inhabited subsets.

1. We show that we can partition Δ^{2n+1} so that the conditions for Sperner's lemma are satisfied. We partition Δ^{2n+1} according to the method in the proof of Sperner's lemma, and label the vertices of simplices constructed by partition of Δ^{2n+1}. Further suppose that we partition Δ^{2n+1} sufficiently fine so that the distance between any pair of vertices of small simplices constructed by partition is sufficiently small. Let K be the set of small simplices constructed by partition of Δ^{2n+1}, and $\mathbf{q} = (q_0, q_1, \dots, q_{2n+1})$ and $\mathbf{q}' = (q'_0, q'_1, \dots, q'_{2n+1})$ be vertices of a simplex of K. Denote the value of Θ at $\mathbf{q}$ by $\Theta(\mathbf{q})$. Let $\theta(\mathbf{q})$ be a point in $\Theta(\mathbf{q})$, and denote the i-th component of $\theta(\mathbf{q})$ by θ_i. Since Θ uniformly has a closed graph, with sufficiently fine partition there exists δ such that if $|\mathbf{q} - \mathbf{q}'| < \delta$, then for $\varepsilon > 0$ $|\theta(\mathbf{q}) - \theta(\mathbf{q}')| < \varepsilon$ for any $\theta(\mathbf{q}) \in \Theta(\mathbf{q})$ and some $\theta(\mathbf{q}') \in \Theta(\mathbf{q}')$, or for some $\theta(\mathbf{q}) \in \Theta(\mathbf{q})$ and any $\theta(\mathbf{q}') \in \Theta(\mathbf{q}')$[2].

   Let $\mathbf{q}^0$ be a vertex of a small $2n+1$-dimensional simplex of K which is labeled with 0 by the labelling method which will be explained below. We take a point $\theta(\mathbf{q}) \in \Theta(\mathbf{q})$ for all other vertices of this simplex so that $|\theta(\mathbf{q}^0) - \theta(\mathbf{q})| < \varepsilon$ is satisfied[3].

   It is important how to label the vertices contained in the faces of Δ^{2n+1}. We label a vertex $\mathbf{q}$ according to the following rule,

   $$\text{If } q_k > \theta_k \text{ or } q_k + \tau > \theta_k, \text{ we label } \mathbf{q} \text{ with } k,$$

   where τ is a positive number. If there are multiple k's which satisfy this condition, we label $\mathbf{q}$ conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

   For example, let $\mathbf{q}$ be a point contained in an $2n$-dimensional face of Δ^{2n+1} such that $q_i = 0$ for one i among $0, 1, 2, \dots, 2n+1$ (the i-th component of its coordinates is 0). With $\tau > 0$, we have $\theta_i > 0$ or $\theta_i < \tau$[4]. When $\theta_i > 0$, from $\sum_{j=0}^{2n+1} q_j = 1$, $\sum_{j=0}^{2n+1} \theta_j = 1$ and $q_i = 0$,

   $$\sum_{j=0, j\neq i}^{2n+1} q_j > \sum_{j=0, j\neq i}^{2n+1} \theta_j.$$

[2] Consider a sequence $(\mathbf{q}_m)_{m\geq 1}$ converging to $\mathbf{q}'$ and a sequence $(\theta(\mathbf{q}_m))_{m\geq 1}$ such that $\theta(\mathbf{q}_m) \in \Theta(\mathbf{q}_m)$ for each m, then closedness of the graph of Θ implies that $(\theta(\mathbf{q}_m))_{m\geq 1}$ converges to a point in $\Theta(\mathbf{q}')$. Similarly, consider a sequence $(\mathbf{q}'_m)_{m\geq 1}$ converging to $\mathbf{q}$ and a sequence $(\theta(\mathbf{q}'_m))_{m\geq 1}$ such that $\theta(\mathbf{q}'_m) \in \Theta(\mathbf{q}'_m)$ for each m, then closedness of the graph of Θ implies that $(\theta(\mathbf{q}'_m))_{m\geq 1}$ converges to a point in $\Theta(\mathbf{q})$.

[3] There may exist a case such that for any $\delta > 0$ we can not take a point $\theta(\mathbf{q})$ for some vertex $\mathbf{q}$ so that $|\theta(\mathbf{q}^0) - \theta(\mathbf{q})| < \varepsilon$ is satisfied. See Note at the end of this proof about such a case.

[4] In constructive mathematics for any real number x we can not prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$.

Then, for at least one j (denote it by k) we have $q_k > \theta_k$, and we label $\mathbf{q}$ with k, where k is one of the numbers which satisfy $q_k > \theta_k$. Since $\theta_i > q_i$, i does not satisfy this condition. Assume $\theta_i < \tau$. $q_i = 0$ implies $\sum_{j=0, j\neq i}^{2n+1} q_j = 1$. Since $\sum_{j=0, j\neq i}^{2n+1} \theta_j \leq 1$

$$\sum_{j=0, j\neq i}^{2n+1} q_j \geq \sum_{j=0, j\neq i}^{2n+1} \theta_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i}^{2n+1} (q_j + \tau) > \sum_{j=0, j\neq i}^{2n+1} \theta_j.$$

There is at least one $j(\neq i)$ which satisfies $q_j + \tau > \theta_j$. Denote it by k, and we label $\mathbf{q}$ with k. k is one of the numbers other than i such that $q_k + \tau > \theta_k$ is satisfied. i itself satisfies this condition ($q_i + \tau > \theta_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i. We have proved that we can label the vertices contained in an $2n$-dimensional face of Δ^{2n+1} such that $q_i = 0$ for one i among $0, 1, 2, \ldots, 2n + 1$ with the numbers other than i. By similar procedures we can show that we can label the vertices contained in an $2n - 1$-dimensional face of Δ^{2n+1} such that $q_i = 0$ for two i's among $0, 1, 2, \ldots, 2n + 1$ with the numbers other than those i's, and so on.

Consider the case where $q_i = q_{i+1} = 0$. We can see that when $\theta_i > 0$ or $\theta_{i+1} > 0$,

$$\sum_{j=0, j\neq i,i+1}^{2n+1} q_j > \sum_{j=0, j\neq i,i+1}^{2n+1} \theta_j.$$

Then, for at least one j(denote it by k) we have $q_k > \theta_k$, and we label $\mathbf{q}$ with k. On the other hand, when $\theta_i < \tau$ and $\theta_{i+1} < \tau$, we have

$$\sum_{j=0, j\neq i,i+1}^{2n+1} q_j \geq \sum_{j=0, j\neq i,i+1}^{2n+1} \theta_j.$$

Then, for a positive number τ

$$\sum_{j=0, j\neq i,i+1}^{2n+1} (q_j + \tau) > \sum_{j=0, j\neq i,i+1}^{2n+1} \theta_j.$$

Thus, there is at least one $j(\neq i, i+1)$ which satisfies $q_j + \tau > \theta_j$. Denote it by k, and we label $\mathbf{q}$ with k.

Next consider the case where $q_i = 0$ for all i other than $2n+1$. If for some i $\theta_i > 0$, then we have $q_{2n+1} > \theta_{2n+1}$, and label $\mathbf{q}$ with $2n+1$. On the other hand, if $\theta_j < \tau$ for all $j \neq 2n+1$, then we obtain $q_{2n+1} \geq \theta_{2n+1}$. It implies $q_{2n+1} + \tau > \theta_{2n+1}$. Thus, we can label $\mathbf{q}$ with $2n+1$.

Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K.

2. Let $\mathbf{q}^0, \mathbf{q}^1, \ldots$ and $\mathbf{q}^{2n+1}$ be the vertices of a fully labeled simplex. We name these vertices so that $\mathbf{q}^0, \mathbf{q}^1, \ldots, \mathbf{q}^{2n+1}$ are labeled, respectively, with $0, 1, \ldots, 2n+1$. The values of Θ at these vertices are $\Theta(\mathbf{q}^0)$, $\Theta(\mathbf{q}^1), \ldots$ and $\Theta(\mathbf{q}^{2n+1})$. Take points $\theta(\mathbf{q}^0), \theta(\mathbf{q}^1), \ldots$ and $\theta(\mathbf{q}^{2n+1})$ such that $\theta(\mathbf{q}^0) \in \Theta(\mathbf{q}^0)$, $\theta(\mathbf{q}^1) \in \Theta(\mathbf{q}^1), \ldots$ and $\theta(\mathbf{q}^{2n+1}) \in \Theta(\mathbf{q}^{2n+1})$. The i-th components of $\mathbf{q}^0$ and $\theta(\mathbf{q}^0)$ are denoted by $\mathbf{q}_i^0$ and $\theta(\mathbf{q}^0)_i$, and so on.

   By our assumption in (1) of this proof when the distance between $\mathbf{q}^0$ and $\mathbf{q}^1$ ($|\mathbf{q}^0 - \mathbf{q}^1|$) is smaller than δ, the distance between $\theta(\mathbf{q}^0)$ and $\theta(\mathbf{q}^1)$ ($|\theta(\mathbf{q}^0) - \theta(\mathbf{q}^1)|$) is smaller than ε. We can make δ satisfying $\delta < \varepsilon$[5]. Suppose $\tau > 0$. About $\mathbf{q}^0$, from the labeling rules we have $\mathbf{q}_0^0 + \tau > \theta(\mathbf{q}^0)_0$. About $\mathbf{q}^1$, also from the labeling rules we have $\mathbf{q}_1^1 + \tau > \theta(\mathbf{q}^1)_1$ which implies $\mathbf{q}_1^1 > \theta(\mathbf{q}^1)_1 - \tau$. $|\theta(\mathbf{q}^0) - \theta(\mathbf{q}^1)| < \varepsilon$ means $\theta(\mathbf{q}^1)_1 > \theta(\mathbf{q}^0)_1 - \varepsilon$. On the other hand, $|\mathbf{q}^0 - \mathbf{q}^1| < \delta$ means $\mathbf{q}_1^0 > \mathbf{q}_1^1 - \delta$. Thus, from

$$\mathbf{q}_1^0 > \mathbf{q}_1^1 - \delta,\ \mathbf{q}_1^1 > \theta(\mathbf{q}^1)_1 - \tau,\ \theta(\mathbf{q}^1)_1 > \theta(\mathbf{q}^0)_1 - \varepsilon$$

   we obtain

$$\mathbf{q}_1^0 > \theta(\mathbf{q}^0)_1 - \delta - \varepsilon - \tau > \theta(\mathbf{q}^0)_1 - 2\varepsilon - \tau$$

   By similar arguments, for each i other than 0,

$$\mathbf{q}_i^0 > \theta(\mathbf{q}^0)_i - 2\varepsilon - \tau. \tag{3.2}$$

   For $i = 0$ we have $\mathbf{q}_0^0 + \tau > \theta(\mathbf{q}^0)_0$. Then,

$$\mathbf{q}_0^0 > \theta(\mathbf{q}^0)_0 - \tau \tag{3.3}$$

   Adding (3.2) and (3.3) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j\neq k}^{2n+1} \mathbf{q}_j^0 > \sum_{j=0, j\neq k}^{2n+1} \theta(\mathbf{q}^0)_j - 4n\varepsilon - (2n+1)\tau.$$

[5] For example, for $\delta < 1$ and $\varepsilon < 1$, if when $|\mathbf{q}^0 - \mathbf{q}^1| < \delta$ we have $|\theta(\mathbf{q}^0) - \theta(\mathbf{q}^1)| < \varepsilon$, then we have $|\theta(\mathbf{q}^0) - \theta(\mathbf{q}^1)| < \varepsilon$ also when $|\mathbf{q}^0 - \mathbf{q}^1| < \delta\varepsilon < \varepsilon$.

From $\sum_{j=0}^{2n+1} \mathbf{q}_j^0 = 1$, $\sum_{j=0}^{2n+1} \theta(\mathbf{q}^0)_j = 1$ we have $1-\mathbf{q}_k^0 > 1-\theta(\mathbf{q}^0)_k - 4n\varepsilon-(2n+1)\tau$, which is rewritten as

$$\mathbf{q}_k^0 < \theta(\mathbf{q}^0)_k + 4n\varepsilon + (2n+1)\tau.$$

Since (3.2) implies $\mathbf{q}_k^0 > \theta(\mathbf{q}^0)_k - 2\varepsilon - \tau$, we have

$$\theta(\mathbf{q}^0)_k - 2\varepsilon - \tau < \mathbf{q}_k^0 < \theta(\mathbf{q}^0)_k + 4n\varepsilon + (2n+1)\tau.$$

Thus,

$$|\mathbf{q}_k^0 - \theta(\mathbf{q}^0)_k| < 4n\varepsilon + n(2n+1)\tau \tag{3.4}$$

is derived. On the other hand, adding (3.2) from 1 to $2n+1$ yields

$$\sum_{j=1}^{2n+1} \mathbf{q}_j^0 > \sum_{j=1}^{2n+1} \theta(\mathbf{q}^0)_j - 2(2n+1)\varepsilon - (2n+1)\tau.$$

From $\sum_{j=0}^{2n+1} \mathbf{q}_j^0 = 1$, $\sum_{j=0}^{2n+1} \theta(\mathbf{q}^0)_j = 1$ we have

$$1 - \mathbf{q}_0^0 > 1 - \theta(\mathbf{q}^0)_0 - 2(2n+1)\varepsilon - (2n+1)\tau. \tag{3.5}$$

Then, from (3.3) and (3.5)

$$|\mathbf{q}_0^0 - \theta(\mathbf{q}^0)_0| < 2(2n+1)\varepsilon + (2n+1)\tau. \tag{3.6}$$

Since n is finite, redefining $2(2n+1)\varepsilon + (2n+1)\tau$ as ε, (3.4) and (3.6) yield

$$|\mathbf{q}_i^0 - \theta(\mathbf{q}^0)_i| < \varepsilon \text{ for all } i. \tag{3.7}$$

Appropriately selecting points $\theta(\mathbf{q}^0)$, $\theta(\mathbf{q}^1)$, … and $\theta(\mathbf{q}^{2n+1})$, every point contained in the fully labeled simplex of K can be made satisfy (3.7).

Let $\mathbf{q}$ be $(\mathbf{p}, \mathbf{z})$, and Θ be g in (3.1). Denote one of the points which satisfy (3.7) by $(\mathbf{p}^*, \mathbf{z}^*)$. Then, for each $\varepsilon > 0$

$$|\varphi_i - p_i^*| < \varepsilon \text{ for all } i, \tag{3.8}$$

and

$$|F(\mathbf{p}^*) - \mathbf{z}^*| < \varepsilon$$

hold. Let $\mathbf{p}^* = (p_0^*, p_1^*, \dots, p_n^*)$, $\mathbf{z}^* = (z_0^*, z_1^*, \dots, z_n^*)$. Then,

$$\left| \frac{p_i^* + \max(z_i^*, 0)}{1 + \sum_{j=0}^n \max(z_j^*, 0)} - p_i^* \right| = \left| \frac{\max(z_i^*, 0) - p_i^* \sum_{j=0}^n \max(z_j^*, 0)}{1 + \sum_{j=0}^n \max(z_j^*, 0)} \right| < \varepsilon.$$

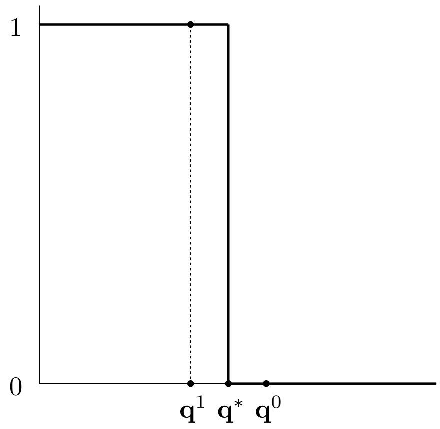

Figure 3: A multi-function in 1-dimensional case

Let $\sum_{j=0}^{n} \max(z_j^*, 0) = \lambda$. We have

$$|\max(z_i^*, 0) - \lambda p_i^*| < (1 + \lambda)\varepsilon.$$

This means

$$-(1 + \lambda)\varepsilon + \lambda p_i^* < \max(z_i^*, 0) < (1 + \lambda)\varepsilon + \lambda p_i^*. \tag{3.9}$$

By $\sum_{i=0}^{n} p_i^* = 1$ there exists k which satisfies $p_k^* > 0$. If for some k satisfying $p_k^* > 0$ we have $z_k^* > 0$, then the weak Walras Law is violated because p_i can not be negative, and $p_k^* z_k^* > 0$ can not be canceled out. Thus, λ as well as ε must be a positive number which may be arbitrarily small, and since p_i^* is finite, $(1 + \lambda)\varepsilon + \lambda p_i^*$ is a real number which may be arbitrarily small. There exists a number which is only slightly larger than $(1+\lambda)\varepsilon+\lambda p^*{}_i$. Replace $(1+\lambda)\varepsilon+\lambda p_i^*$ by such a number, and denote it by ε. Then, we obtain

$$\max(z_i^*, 0) < \varepsilon. \tag{3.10}$$

This holds for all i. Therefore,

$$\mathbf{z}^* < \varepsilon\mathbf{e}$$

is obtained.

Note

There may exist a case such that for any $\delta > 0$ we can not take a point $\theta(\mathbf{q})$ for some vertex $\mathbf{q}$ so that $|\theta(\mathbf{q}^0) - \theta(\mathbf{q})| < \varepsilon$ is satisfied. An example in a 1-dimensional case is a multi-function from [0, 1] to [0, 1] depicted in Figure 3.

The coordinates of the points 0 and 1 are, respectively, $(0, 1)$ and $(1, 0)$. And coordinates of other points in $[0, 1]$ are similar. Even if $|\mathbf{q}^0 - \mathbf{q}^1| < \delta$ for any $\delta > 0$, $|\theta(\mathbf{q}^0) - \theta(\mathbf{q}^1)| > 0$. $\mathbf{q}^0$ and $\mathbf{q}^1$ are, respectively, numbered with 0 and 1. In such a case we must consider further partition of a simplex $[\mathbf{q}^0, \mathbf{q}^1]$ and take a limit when $\delta \to 0$. At the limit of vertices of a fully labeled simplex $\mathbf{q}^*$ there are points $\theta^1(\mathbf{q}^*) \in F(\mathbf{q}^*)$ and $\theta^2(\mathbf{q}^*) \in F(\mathbf{q}^*)$ such that

$$\mathbf{q}_0^* > \theta^1(\mathbf{q}^*)_0 - \tau \text{ and } \mathbf{q}_1^* > \theta^2(\mathbf{q}^*)_1 - \tau.$$

Since $\mathbf{q}_0^* + \mathbf{q}_1^* = 1$ and $\theta^2(\mathbf{q}^*)_0 + \theta^2(\mathbf{q}^*)_1 = 1$, the latter implies

$$\mathbf{q}_0^* < \theta^2(\mathbf{q}^*)_0 + \tau.$$

Thus,

$$\theta^1(\mathbf{q}^*)_0 - \tau < \mathbf{q}_0^* < \theta^2(\mathbf{q}^*)_0 + \tau.$$

Define a point in $F(\mathbf{q}^*)$ by

$$\theta^*(\mathbf{q}^*) = \alpha\theta^1(\mathbf{q}^*) + (1-\alpha)\theta^2(\mathbf{q}^*),\ 0 \le \alpha \le 1.$$

By the convexity of $F(\mathbf{q}^*)$, $\theta^*(\mathbf{q}^*) \in F(\mathbf{q}^*)$. Let

$$\alpha = \frac{\theta^2(\mathbf{q}^*)_0 + \tau - \mathbf{q}_0^*}{[\theta^2(\mathbf{q}^*)_0 + \tau - \mathbf{q}_0^*] + [\mathbf{q}_0^* - \theta^1(\mathbf{q}^*)_0 + \tau]} = \frac{\theta^2(\mathbf{q}^*)_0 + \tau - \mathbf{q}_0^*}{\theta^2(\mathbf{q}^*)_0 - \theta^1(\mathbf{q}^*)_0 + 2\tau},$$

and

$$1 - \alpha = \frac{\mathbf{q}_0^* - \theta^1(\mathbf{q}^*)_0 + \tau}{\theta^2(\mathbf{q}^*)_0 - \theta^1(\mathbf{q}^*)_0 + 2\tau}.$$

Then,

$$\theta^*(\mathbf{q}^*)_0 = \frac{\theta^1(\mathbf{q}^*)_0(\tau - \mathbf{q}_0^*) + \theta^2(\mathbf{q}^*)_0(\tau + \mathbf{q}_0^*)}{\theta^2(\mathbf{q}^*)_0 - \theta^1(\mathbf{q}^*)_0 + 2\tau}.$$

And so we have

$$\mathbf{q}_0^* - \theta^*(\mathbf{q}^*)_0 = \frac{\tau[2\mathbf{q}_0^* - \theta^1(\mathbf{q}^*)_0 - \theta^2(\mathbf{q}^*)_0]}{\theta^2(\mathbf{q}^*)_0 - \theta^1(\mathbf{q}^*)_0 + 2\tau}.$$

Since τ may be arbitrarily small, for any $\varepsilon > 0$ we obtain

$$|\mathbf{q}_0^* - \theta^*(\mathbf{q}^*)_0| < \varepsilon.$$

Similarly

$$|\mathbf{q}_1^* - \theta^*(\mathbf{q}^*)_1| < \varepsilon$$

is derived.

A case of higher dimension is similar.

We have completed the proof of the approximate version of the Gale-Nikaido lemma. ■

If we interpret p_i and z_i be the price and excess demand of each good, and $F(\mathbf{p})$ be a multi-valued excess demand function, then this theorem implies the existence of an approximate equilibrium of a competitive exchange economy with multi-valued excess demand functions at which excess demand for each good is smaller than ε.

4. From the approximate version of Gale-Nikaido lemma to Sperner's lemma

In this section we will derive Sperner's lemma from an approximate version of the Gale-Nikaido lemma. Let partition an n-dimensional simplex Δ^n. Denote the set of small n-dimensional simplices of Δ^n constructed by partition by K. Vertices of these small simplices of K are labeled with the numbers 0, 1, 2, ..., n similarly to the proof of Sperner's lemma. Denote vertices of an n-dimensional simplex of K by $x^0, x^1, \dots, x^n$, the j-th component of x^i by x^i_j, and the label of x^i by $l(x^i)$. Let τ be a positive number which is smaller than $x^i_{l(x^i)}$ for all x^i, and define a function $f(x^i)$ as follows[6];

$$f(x^i) = (f_0(x^i), f_1(x^i), \dots, f_n(x^i)),$$

and

$$f_j(x^i) = \begin{cases} x^i_j - \tau & \text{for } j = l(x^i), \\ x^i_j + \dfrac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases} \tag{4.1}$$

f_j denotes the j-th component of f. From the labeling rules $x^i_{l(x^i)} > 0$ for all x^i, and so $\tau > 0$ is well defined. Since $\sum_{j=0}^{n} f_j(x^i) = \sum_{j=0}^{n} x^i_j = 1$, we have

$$f(x^i) \in \Delta^n.$$

We extend f to all points in the simplex by convex combinations of its values on the vertices of the simplex. Let y be a point in an n-dimensional simplex of K whose vertices are $x^0, x^1, \dots, x^n$. Then, y and $f(y)$ are represented as follows;

$$y = \sum_{i=0}^{n} \lambda_i x^i, \text{ and } f(y) = \sum_{i=0}^{n} \lambda_i f(x^i), \ \lambda_i \geq 0, \ \sum_{i=0}^{n} \lambda_i = 1.$$

[6]We refer to [10] about the definition of this function.

Let us show that f is uniformly continuous. Let y and y' be distinct points in the same small n-dimensional simplex of K. They are represented as

$$y = \sum_{i=0}^{n} \lambda_i x^i, \ y' = \sum_{i=0}^{n} \lambda'_i x^i,$$

and so

$$y - y' = \sum_{i=0}^{n} (\lambda_i - \lambda'_i) x^i \text{ and } y_j - y'_j = \sum_{i=0}^{n} (\lambda_i - \lambda'_i) x^i_j \text{ for each } j.$$

Then, we have

$$f(y) - f(y') = \sum_{i=0}^{n} (\lambda_i - \lambda'_i) f(x^i)$$

and for each j

$$\begin{aligned} f_j(y) - f_j(y') &= \sum_{i=0}^{n} (\lambda_i - \lambda'_i) x^i_j + \sum_{i:j \neq l(i)} (\lambda_i - \lambda'_i) \frac{\tau}{n} - \sum_{i:j=l(i)} (\lambda_i - \lambda'_i) \tau \\ &= y_j - y'_j + \sum_{i:j \neq l(i)} (\lambda_i - \lambda'_i) \frac{\tau}{n} - \sum_{i:j=l(i)} (\lambda_i - \lambda'_i) \tau \end{aligned}$$

Since τ is finite, appropriately selecting λ'_i given λ_i for each i we can make $|f_j(y) - f_j(y')|$ sufficiently small corresponding to the value of $|y_j - y'_j|$ for each j, and so make $|f(y) - f(y')|$ sufficiently small corresponding to the value of $|y - y'|$. Thus, f is uniformly continuous.

Now, using f, we construct a function $F(x) = \mathbf{z} = \{z_0, z_1, \ldots, z_n\}$ such that

$$z_i = f_i(x) - x_i \mu(x), \ i = 0, 1, \ldots, n. \tag{4.2}$$

$x \in \Delta^n$ and $\mu(x)$ is defined by

$$\mu(x) = \frac{\sum_{i=0}^{n} x_i f_i(x)}{\sum_{i=0}^{n} x_i^2}.$$

Each $z_i(x)$ is uniformly continuous, and satisfies the Weak Walras law as shown below. Multiplying x_i to (4.2) for each i, and adding them from 0 to n yields

$$\begin{aligned} \sum_{i=0}^{n} x_i z_i &= \sum_{i=0}^{n} x_i f_i(x) - \mu(x) \sum_{i=0}^{n} x_i^2 = \sum_{i=0}^{n} x_i f_i(x) - \frac{\sum_{i=0}^{n} x_i f_i(x)}{\sum_{i=0}^{n} x_i^2} \sum_{i=0}^{n} x_i^2 \\ &= \sum_{i=0}^{n} x_i f_i(x) - \sum_{i=0}^{n} x_i f_i(x) = 0. \end{aligned} \tag{4.3}$$

Now define the following function.

$$g(x, \mathbf{z}) = \varphi(x, \mathbf{z}) \times F(x),$$

where

$$\varphi(x, \mathbf{z}) = (\varphi_0, \varphi_1, \ldots, \varphi_n), \ \varphi_i(x, \mathbf{z}) = \frac{x_i + \max(z_i, 0)}{1 + \sum_{j=0}^{n} \max(z_j, 0)}.$$

g is a uniformly continuous function of $(x, \mathbf{z})$, and it is a special case of a compact and convex valued multi-function with uniformly closed graph. Therefore, it satisfies the conditions for the approximate version of the Gale-Nikaido lemma. Then, there exist x^* and $\mathbf{z}^*$ such that

$$|F(x^*) - \mathbf{z}^*| < \varepsilon, \ \text{and } \mathbf{z}^* < \varepsilon\mathbf{e}.$$

From $\max(z_i, 0) < \varepsilon$ (see (3.10)) we have $f_i(x^*) - x_i^*\mu(x^*) < \varepsilon$ for all i with $\varepsilon > 0$. Since it is impossible that $z_i < 0$ for i satisfying $x_i^* > 0$ because of (4.3), we have $z_i = f_i(x^*) - x_i^*\mu(x^*) > -\varepsilon$ for such i. Also for i such that $x_i^* < \varepsilon$, we have $z_i = f_i(x^*) - x_i^*\mu(x^*) > -\varepsilon$. Therefore,

$$-\varepsilon < f_i(x^*) - x_i^*\mu(x^*) < \varepsilon \tag{4.4}$$

is obtained. Adding this inequality side by side from 0 to n yields

$$-(n+1)\varepsilon < \sum_{i=0}^{n} f_i(x^*) - \mu(x^*) \sum_{i=0}^{n} x_i^* < (n+1)\varepsilon.$$

From $\sum_{i=0}^{n} f_i(x^*) = \sum_{i=0}^{n} x_i^* = 1$ we obtain

$$1 - (n+1)\varepsilon < \mu(x^*) < 1 + (n+1)\varepsilon. \tag{4.5}$$

Further from (4.4) and (4.5) we get

$$x_i^* - (n+1)\varepsilon x_i^* - \varepsilon < f_i(x^*) < x_i^* + (n+1)\varepsilon x_i^* + \varepsilon.$$

Since n and x_i^* are finite, redefining $(n+1)\varepsilon x_i^* + \varepsilon$ as ε, we have $-\varepsilon < f_i(x^*) - x_i^* < \varepsilon$, that is,

$$|f_i(x^*) - x_i^*| < \varepsilon$$

is derived. This relation holds for all i.

Let $\gamma > 0$ and $\tilde{x}$ be a point in $V(x^*, \gamma)$, where $V(x^*, \gamma)$ is a γ-neighborhood of x^*. If γ is sufficiently small, uniform continuity of f means

$$|f_i(\tilde{x}) - \tilde{x}_i| < \varepsilon \tag{4.6}$$

for any $\varepsilon > 0$ and for all i. $\tilde{x}_i$ is the i-th component of $\tilde{x}$. Let Δ^{n^*} be a simplex of K which contains $\tilde{x}$, and $x^0, x^1, \dots, x^n$ be the vertices of Δ^{n^*}. Then, $\tilde{x}$ and $F(\tilde{x})$ are represented as

$$\tilde{x} = \sum_{i=0}^{n} \lambda_i x^i \text{ and } f(\tilde{x}) = \sum_{i=0}^{n} \lambda_i f(x^i),\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1.$$

(4.1) implies that if only one x^k among $x^0, x^1, \dots, x^n$ is labeled with i, we have

$$|f_i(\tilde{x}) - \tilde{x}_i| = \left| \sum_{j=0}^{n} \lambda_j x_i^j + \sum_{j=0, j\neq k}^{n} \lambda_j \frac{\tau}{n} - \lambda_k \tau - \tilde{x}_i \right| = \left| \left(\frac{1}{n} \sum_{j=0, j\neq k}^{n} \lambda_j - \lambda_k \right) \tau \right| < \varepsilon.$$

x_i^j is the i-th component of x^j. This means

$$\frac{1}{n} \sum_{j=0, j\neq k}^{n} \lambda_j - \lambda_k \approx 0.$$

It is satisfied with $\lambda_k \approx \dfrac{1}{n+1}$ for all k. On the other hand, if no x^j is labeled with i, we have

$$f_i(\tilde{x}) = \sum_{j=0}^{n} \lambda_j x_i^j = x_i^* + \left(1 + \frac{1}{n}\right) \tau,$$

and then (4.6) can not be satisfied. Thus, for each i one and only one x^j must be labeled with i. Therefore, Δ^{n^*} must be a fully labeled simplex. We have completed the proof of Sperner's lemma.

5. Concluding Remarks

In this paper we have presented a proof of the existence of an approximate equilibrium of an competitive economy with multi-valued demand and supply functions by Sperner's lemma from the viewpoint of constructive mathematics. We are studying some related problems such as the existence of an approximate Nash equilibrium in a finite strategic game with multi-valued best responses, a constructive version of the Fan-Glicksberg fixed point theorem for multi-functions in a locally convex space and its application to a proof of the existence of an approximate Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions.

Appendix

Proof of Sperner's lemma

We prove this lemma by induction about the dimension of Δ^n. When $n = 0$, we have only one point with the number 0. It is the unique 0-dimensional simplex. Therefore

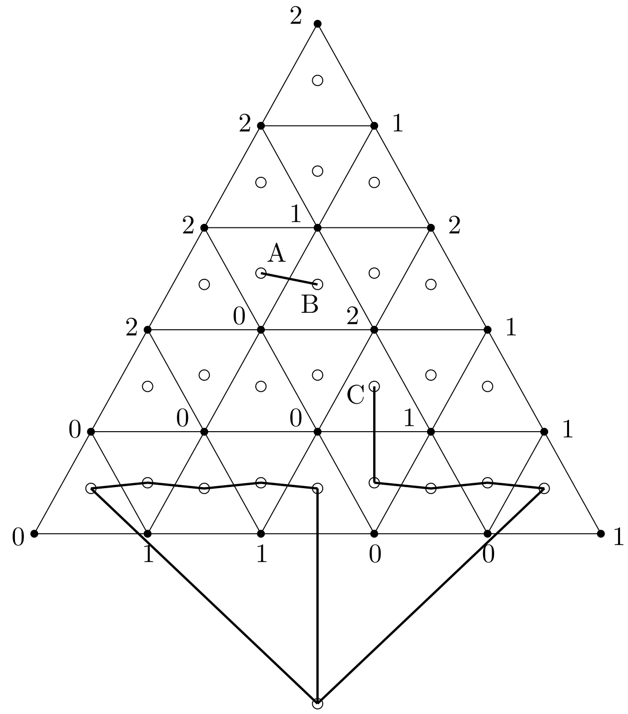

Figure 4: Sperner's lemma

the lemma is trivial. When $n = 1$, a partitioned 1-dimensional simplex is a segmented line. The endpoints of the line are labeled distinctly, with 0 and 1. Hence in moving from endpoint 0 to endpoint 1 the labeling must switch an odd number of times, that is, an odd number of edges labeled with 0 an 1 may be located in this way.

Next consider the case of 2 dimension. Assume that we have partitioned a 2-dimensional simplex (triangle) Δ^n as explained above. Consider the face of Δ^n labeled with 0 and 1[7]. It is the base of the triangle in Figure 4. Now we introduce a dual graph that has its nodes in each small triangle of K plus one extra node outside the face of Δ^n labeled with 0 and 1 (putting a dot in each small triangle, and one dot outside Δ^n). We define edges of the graph that connect two nodes if they share a side labeled with 0 and 1. See Figure 4. White circles are nodes of the graph, and thick lines are its edges. Since from the result of 1-dimensional case there are an odd number of faces of K labeled with 0 and 1 contained in the face of Δ^n labeled with 0 and 1, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the triangle which has odd degree. Each node of our graph except for the outside node is contained in one of small triangles

[7]We call edges of triangle Δ^n *faces* to distinguish between them and edges of a dual graph which we will consider later.

of K. Therefore, if a small triangle of K has one face labeled with 0 and 1, the degree of the node in that triangle is 1; if a small triangle of K has two such faces, the degree of the node in that triangle is 2, and if a small triangle of K has no such face, the degree of the node in that triangle is 0. Thus, if the degree of a node is odd, it must be 1, and then the small triangle which contains this node is labeled with 0, 1 and 2 (fully labeled). In Figure 4 triangles which contain one of the nodes A, B, C are fully labeled triangles.

Now assume that the theorem holds for dimensions up to $n-1$. Assume that we have partitioned an n-dimensional simplex Δ^n. Consider the fully labeled face of Δ^n which is a fully labeled $n-1$-dimensional simplex. Again we introduce a dual graph that has its nodes in small n-dimensional simplices of K plus one extra node outside the fully labeled face of Δ^n (putting a dot in each small n-dimensional simplex, and one dot outside Δ^n). We define the edges of the graph that connect two nodes if they share a face labeled with $0, 1, \dots, n-1$. Since from the result of $n-1$-dimensional case there are an odd number of fully labeled faces of small simplices of K contained in the $n-1$-dimensional fully labeled face of Δ^n, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since, by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the simplex which has odd degree. Each node of our graph except for the outside node is contained in one of small n-dimensional simplices of K. Therefore, if a small simplex of K has one fully labeled face, the degree of the node in that simplex is 1; if a small simplex of K has two such faces, the degree of the node in that simplex is 2, and if a small simplex of K has no such face, the degree of the node in that simplex is 0. Thus, if the degree of a node is odd, it must be 1, and then the small simplex which contains this node is fully labeled.

> If the number (label) of a vertex other than vertices labeled with $0, 1, \dots, n-1$ of an n-dimensional simplex which contains a fully labeled $n-1$-dimensional face is n, then this n-dimensional simplex has one such face, and this simplex is a fully labeled n-dimensional simplex. On the other hand, if the number of that vertex is other than n, then the n-dimensional simplex has two such faces.

We have completed the proof of Sperner's lemma.

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