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A proof of the existence of approximate core in NTU game directly by Sperner's lemma: A constructive analysis[1]

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Abstract

It is often said that Brouwer's fixed point theorem can not be constructively proved. Thus, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) with closed graph and the existence of a core in an NTU (non-transferable utility) game also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. It seems that we can also constructively prove a constructive version of Kakutani's fixed point theorem, and using it we can prove the existence of an approximate core in an NTU game. Then, can we prove the existence of an approximate core in an NTU game directly by Sperner's lemma? We present such a proof. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

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1. Introduction

It is often said that Brouwer's fixed point theorem can not be constructively proved.

> [5] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics a la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [2] or [4]. Brouwer's fixed point theorem can be constructively, in the sense of constructive mathematics a la Bishop, proved only approximately. The existence of an exact fixed point of a function which satisfies some property of local non-constancy may be constructively proved.

Thus, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) with closed graph and the existence of a core in an NTU (non-transferable utility) game also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. See [4] and [8]. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. It seems that we can also constructively prove a constructive version of Kakutani's fixed point theorem, and using it we can prove the existence of an approximate core in an NTU game.

Then, can we prove the existence of an approximate core in an NTU game directly by Sperner's lemma?

We present such a proof. We will show that there exists an approximate core in any *approximately balanced* NTU game by Sperner's lemma. The definitions of an approximately balanced game and an approximate core will be presented in Section 3.

In the next section we prove Sperner's lemma. This proof is a standard constructive proof. In Section 3 we will show the existence of an approximate core in an approximately balanced game by Sperner's lemma. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

2. Sperner's lemma

To prove Sperner's lemma we use the following simple result in graph theory, Handshaking lemma. A *graph* refers to a collection of vertices and a collection of edges that connect pairs of vertices. Each graph may be undirected or directed. Figure 1 is an example of an undirected graph. The degree of a vertex of a graph is defined to be the number of edges incident to the vertex, with loops counted twice. Each vertex has odd degree or even degree. Let v denote a vertex and V denote the set of all vertices.

Lemma 2.1. [Handshaking lemma] Every undirected graph contains an even number of vertices of odd degree. That is, the number of vertices that have an odd number of incident edges must be even.

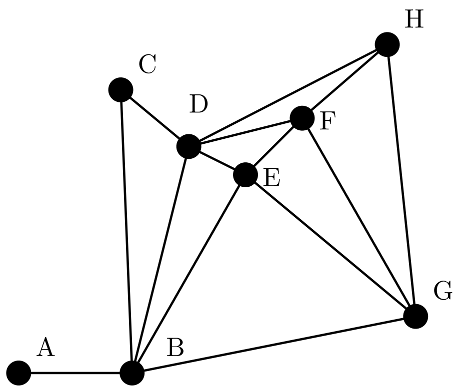

Figure 1: Example of graph

It is a simple lemma. But for completeness of arguments we provide a proof.

Proof. Prove this lemma by double counting. Let $d(v)$ be the degree of vertex v. The number of vertex-edge incidences in the graph may be counted in two different ways: by summing the degrees of the vertices, or by counting two incidences for every edge. Therefore,

$$\sum_{v\in V} d(v) = 2e,$$

where e is the number of edges in the graph. The sum of the degrees of the vertices is therefore an even number. It could happen if and only if an even number of the vertices had odd degree. ■

Let Δ denote an n-dimensional simplex. n is a positive integer at least 2. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Figure 2 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. The vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with $2, \dots,$ a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate $(k = 0, 1, \dots, n)$ is 1 and all other coordinates are 0 is labeled with k.

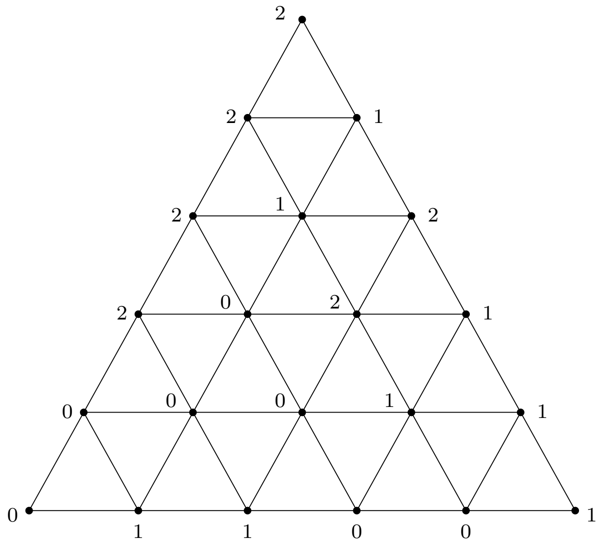

Figure 2: Partition and labeling of 2-dimensional simplex

2. If a vertex of simplices of K is contained in an $n-1$-dimensional face of Δ, then that vertex is labeled with some number which is the same as the number of a vertex of that face.

3. If a vertex of simplices of K is contained in an $n-2$-dimensional face of Δ, then that vertex is labeled with some number which is the same as the number of a vertex of that face. And similarly for cases of lower dimension.

4. A vertex contained in inside of Δ is labeled with arbitrary number among 0, 1, $\dots, n$.

A small simplex of K which is labeled with the numbers $0, 1, \dots, n$ is called a *fully labeled simplex*. Now let us prove Sperner's lemma.

Lemma 2.2. [Sperner's lemma] If we label the vertices of simplices of K following above rules $1 \sim 4$, then there are an odd number of fully labeled simplices. Thus, there exists at least one fully labeled simplex.

Proof. Our proof is standard. But again for completeness of arguments we provide a proof in Appendix 4. ■

Since n and partition of Δ are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n-dimensional simplex of K through finite steps.

3. Approximate core in an NTU game

In this section we consider a core in an NTU (non-transferable utility) game. The model of an NTU game is as follows. There are $n (n \geq 2)$ players. Denote the set of players by N, which is a finite set. Denote the set of n-dimensional vectors of players' payoffs, which can be realized by a coalition of players S, by $V(S) \in R^n$. For the coalition of all players the set is $V(N)$, and the set for one player i is $V(\{i\})$. Denote vectors in $V(S)$ by u, v and so on, and denote their components by u_i, v_i and so on. About $V(S)$ we assume

Assumption 1 For each S (including the cases of $S = N$ and $S = \{i\}$), $V(S)$ is a bounded from above and inhabited (nonempty) closed set of R^n. Boundedness means that there exists a real number q such that if $x \in V(S)$ and $u_S \geq 0$ ($u_i \geq 0$ for all $i \in S$), then $u_i < q$ for all $i \in S$.

Assumption 2 $V(S)$ is *comprehensive* in the sense that if $u \in V(S)$ and $v \leq u$ ($v_i \leq u_i$ for all $i \in N$), then $v \in V(S)$.

Assumption 3 $V(S)$ is *cylindrical* in the sense that if $u \in V(S)$ and $u_i = v_i$ for all $i \in S$, then $v \in V(S)$.

Assumption 4 There exists a vector $b \gg 0$ ($b_i > 0$ for all $i \in N$) such that for each i, $V(\{i\}) = \{u \in R^n | u_i \leq b_i\}$.

We define an *approximate core* as follows.

Approximate core If, for a vector $u \in V(N)$ and a coalition S there exists another vector $v \in V(S)$ such that $v_i > u_i$ for all $i \in S$, then we say that a coalition S *improves* u through v. Let $u^* \in V(N)$ be a vector which is not improved by any coalition within the range of ε. Then, for $\varepsilon > 0$ and some i,

$$u_i^* > v_i - \varepsilon \text{ for each } v_i \text{ in all } S \subset N. \tag{3.1}$$

We call the set of vectors which satisfy (3.1) an *approximate core*.

Next we define an *approximately balanced game*. Let $|S|$ denote the number of players in S, and m^S be an n-dimensional vector such that $m_i^S = \frac{1}{|S|}$ for all $i \in S$ and $m_i^S = 0$ for all other i. m^N is a vector whose each component is $\frac{1}{n}$. Let F be a collection of subsets of N. If there exists a positive real number λ_S for each $S \in F$ which satisfies

$$\sum_{S \in F} \lambda_S m^S = m^N,$$

then we say that F is a *balanced collection of sets*. On the other hand, if for $\varepsilon > 0$,

$$\left| \sum_{S \in F} \lambda_S m^S - m^N \right| < \varepsilon \tag{3.2}$$

is satisfied, then we say that F is an *approximately balanced collection of sets*. For each component i we have $\left|\sum_{S\in F,\ i\in S}\frac{\lambda_S}{|S|}-\frac{1}{n}\right|<\varepsilon$, and so (redefining $n\varepsilon$ as ε)

$$\left|\sum_{S\in F,\ i\in S}\frac{n}{|S|}\lambda_S-1\right|<\varepsilon$$

is obtained. Thus, for each i the sum of $\frac{n}{|S|}\lambda_S$ for all coalitions S which include i approximately equals one. Since the number of players in S is $|S|$, the sum of $\sum_{S\in F,\ i\in S}\frac{n}{|S|}\lambda_S$ for all payers is $\sum_{S\in F}n\lambda_S$, and it approximately equals n. Therefore, we have $\left|\sum_{S\in F}n\lambda_S-n\right|<n\varepsilon$, and so

$$\left|\sum_{S\in F}\lambda_S-1\right|<\varepsilon.$$

That is, if F is an approximately balanced collection of sets, the sum of λ_S for all $S\in F$ approximately equals one.

Approximately balanced game If in an NTU game

$$\bigcap_{S\in F}V(S)\subset V(N)$$

holds for all approximately balanced collections of sets F, then we say that this game is an *approximately balanced game*.

Let $Q=\{x\in R^n|x_i\le q,\ \text{for all } i\}$ and define the following set

$$W=\left(\bigcup_{S\subset N}V(S)\right)\cap Q.$$

Since $V(S)$ and Q are comprehensive, W is also comprehensive. Let ∂W be the boundary of W. Then, about W and ∂W we obtain the following results.

Lemma 3.1.

1. If for two vectors x and y, $x\in\partial W$ and $y_i>x_i$ for all i, then y is not included in ∂W.

2. If $x \in \partial W$ and $x_j < \varepsilon$ for some j for $\varepsilon > 0$, then $x_i > q - \tau$ for $\tau > 0$ and some $i(\neq j)$.

Proof.

1. It is clear because ∂W is the boundary of W.

2. By Assumption 3 and 4 $V(\{j\})$ includes a vector such that $x_j > 0$ and all other components equal q. Therefore, by (1) of this lemma any vector such that $x_j < \varepsilon$ and all other components are smaller than q is not included in ∂W. ■

The metric complement $-C$ of a set C, which is a subset of a metric space X, is defined as follows.

$$-C = \{x \in X : \rho(x, C) > 0\},$$

where $\rho(x, C)$ is the distance between x and C. On the other hand, the complement $\sim C$ of C is defined as follows.

$$\sim C = \{x \in X : \forall y \in C(x \neq y)\}.$$

It is clear that $-C \subset \sim C$. In [3] it was shown that if C is an inhabited open convex subset of a normed space, $-C$ is dense in $\sim C$. The boundary ∂C of C is defined as follows:

$$\partial C = \overline{C} \cup \overline{\sim C}.$$

$\overline{C}$ and $\overline{\sim C}$ are, respectively, closures of C and $\sim C$. In [3] the following theorem has been constructively proved.

Theorem 3.2. [Proposition 5.1.5 in [3]] Let C be an open convex subset of a Banach space X such that $C \cup -C$ is dense in X, and let $\xi \in C$. For each $z \in -C$ and each $t \in [0, 1]$ write

$$z_t = t\xi + (1 - t)z.$$

Then the following hold:

1. $\gamma(\xi, z) = \inf\{t \in [0, 1] : z_t \in C\}$ exists, and $0 < \gamma(\xi, z) < 1$.

2. $z_{\gamma(\xi,z)}$ is the unique intersection of $[\xi, z]$ with ∂C.

3. If $\gamma(\xi, z) < t \leq 1$, then $z_t \in C$.

4. If $0 \leq t < \gamma(\xi, z)$, then $z_t \in -C$.

Moreover, the mapping $(\xi, z) \longrightarrow z_{\gamma(\xi,z)}$ of $C \times -C$ into ∂C is uniformly continuous in $C \times -C$.

With these preliminaries we show the following theorem.

Theorem 3.3. An approximately balanced game has an approximate core.

Proof.

1. By Theorem 3.2 there exists the following single-valued uniformly continuous function.

$$f(x) = \{y \in \partial W | y = tx,\ x \in \Delta^{n-1} \text{ for some } t \geq 0\}.$$

y is a point on the line from a point in an $n-1$-dimensional simplex Δ^{n-1} through ∂W. Although W is not an open set, the interior W° of W is open and $\partial W = \overline{W^\circ} \cap \overline{\sim W^\circ}$. Let

$$z_\alpha = \alpha\xi + (1-\alpha)t\xi,\ t > 0,$$

where ξ is a point in W° and $t\xi$ is a point in $-W^\circ$. Although W may not be convex. the convexity of the following set T suffices for application of Theorem 3.2 to our case.

$$T = \{\alpha \in [0, 1] :\ z_\alpha \in W^\circ\}.$$

If a point tx is in W°, the comprehensiveness of W implies that any point $t'x$ such that $0 \leq t' < t$ is included in W°. Thus, T is convex.

We define the following multi-function $G :\ \Delta^{n-1} \to 2^{\Delta^{n-1}}$.

$$G(x) = \{m^S | f(x) \in V(S)\}.$$

We write $f(x) \in V(S)$ because $f(x)$ is a single-valued function. Since $f(x)$ is included in ∂W, it is also included in $\bigcup_{S \subset N} V(S)$. Therefore, it is included in some $V(S)$, and hence $G(x)$ is inhabited.

> A graph of a multi-function G is
>
> $$\Gamma(G) = \cup_{x \in X}\{x\} \times G(x).$$
>
> If $\Gamma(G)$ is a closed set, we say that G has a closed graph.

It implies the following fact.

> Consider sequences $(x(n))_{n\geq 1}$ and $(y(n))_{n\geq 1}$ such that $y(n) \in G(x(n))$. If $x(n) \longrightarrow x$ and $y(n) \longrightarrow y$, then $y \in G(x)$.

According to [3] this means

> If for each neighborhood $U(x, \varepsilon)$ of x there exists n_0 such that $x(n) \in U(x, \varepsilon)$ when $n \geq n_0$, then for the union of neighborhoods $\cup_{y \in G(x)} V(y, \varepsilon)$ of points in $G(x)$ there exists n_0' such that $y(n) \in \cup_{y \in G(x)} V(y, \varepsilon)$ when $n \geq n_0'$.

Further we consider a uniform version of this property for multi-functions, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. It means that n_0 and n'_0 depend on only ε not on x.

We show that $G(x)$ uniformly has a closed graph. Since $f(x)$ is uniformly continuous, for $\varepsilon > 0$ and each x and x' we can select δ such that if $|x - x'| < \delta$, then $|f(x) - f(x')| < \varepsilon$. Therefore, for a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_i > \ldots$ we can select a sequence of points $x^1, x^2, \ldots, x^i, \ldots$ and a sequence of positive numbers $\delta_1 > \delta_2 > \cdots > \delta_i > \ldots$ such that if $|x - x^i| < \delta_i$, then $|f(x) - f(x^i)| < \varepsilon_i$. Thus, $|f(x) - f(x^i)| < \varepsilon_{i_0}$ holds for $i > i_0$ with some i_0, and hence a sequence of points $f(x^1), f(x^2), \ldots, f(x^i), \ldots$ converges to $f(x)$. If $f(x^i) \in V(S)$, that is, $m^S \in G(x^i)$ for some S for all x^i, we have $f(x) \in V(S)$ because $V(S)$ is a closed set. Thus, $m^S \in G(x)$, and $G(x)$ uniformly has a closed graph.

Let $co(G)$ be the convex-hull of vectors in $G(x)$. Since $G(x)$ has a uniformly closed graph, $co(G)$ also uniformly has a closed graph. Let $m^{S_0}, m^{S_1}, \ldots, m^{S_m}$ be vectors in $G(x)$, a vector included in $co(G)$ is represented as follows,

$$v = \sum_{j=0}^{m} \mu_j m^{S_j}, \ \sum_{j=0}^{m} \mu_i = 1, \ \mu_i \geq 0$$

For all $x, g \in \Delta^{n-1}$ we define a function h by

$$h_i(x, g) = \frac{x_i + \max\left(g_i - \frac{1}{n}, 0\right)}{1 + \sum_{j=0}^{n} \max\left(g_j - \frac{1}{n}, 0\right)} \text{ for each } i$$

$\sum_{i=0}^{n} x_i = 1$ means

$$\sum_{i=0}^{n} h_i(x, g) = \frac{\sum_{i=0}^{n} x_i + \sum_{i=0}^{n} \max\left(g_i - \frac{1}{n}, 0\right)}{1 + \sum_{j=0}^{n} \max\left(g_j - \frac{1}{n}, 0\right)} = 1.$$

Thus, $h(x, g) = (h_0, h_1, \ldots, h_n)$ is a uniformly continuous function from $\Delta^{n-1} \times \Delta^{n-1}$ to Δ^{n-1}. Then, a multi-function $h \times co(G)$ defined by

$$h \times co(G) : \ \Delta^{n-1} \times \Delta^{n-1} \to \Delta^{n-1} \times 2^{\Delta^{n-1}}$$

is convex-valued and has a uniformly closed graph. $\Delta^{n-1} \times \Delta^{n-1}$ is homeomorphic to $2(n-1)$-dimensional simplex.

2. Let $\Phi(y) = h \times co(G)(y) : \Delta^{n-1} \times \Delta^{n-1} \to \Delta^{n-1} \times 2^{\Delta^{n-1}}$ with $y = (x, g)$. We show that we can partition $\Delta^{2(n-1)}$ so that the conditions for Sperner's lemma are

satisfied. We partition $\Delta^{2(n-1)}$ according to the method in the proof of Sperner's lemma, and label the vertices of simplices constructed by partition of $\Delta^{2(n-1)}$. Further suppose that we partition $\Delta^{2(n-1)}$ sufficiently fine so that the distance between any pair of vertices of small simplices is sufficiently small. Let K be the set of small simplices constructed by partition of $\Delta^{2(n-1)}$, $y = (y_0, y_1, \dots, y_{2(n-1)})$ and $y' = (y'_0, y'_1, \dots, y'_{2(n-1)}$ be vertices of a simplex of K. Denote the value of Φ at y by $\Phi(y)$. Let $\varphi(y)$ be a point in $\Phi(y)$, and denote the i-th component of $\varphi(y)$ by φ_i. Since Φ has a uniformly closed graph, with sufficiently fine partition there exists δ such that if $|y - y'| < \delta$, then for $\varepsilon > 0$ $|\varphi(y) - \varphi(y')| < \varepsilon$ for any $\varphi(y) \in \Phi(y)$ and some $\varphi(y') \in \Phi(y')$, or for some $\varphi(y) \in \Phi(y)$ and any $\varphi(y') \in \Phi(y')$[2]. Let y^0 be a vertex of a small $2(n-1)$-dimensional simplex of K which is labeled with 0 by the labelling method which will be explained below. We take a point $\varphi(y) \in \Phi(y)$ for all other vertices of this simplex so that $|\varphi(y^0) - \varphi(y)| < \varepsilon$ is satisfied[3]. It is important how to label the vertices contained in the faces of $\Delta^{2(n-1)}$. We label a vertex y according to the following rule,

$$\text{If } y_k > \varphi_k \text{ or } y_k + \tau > \varphi_k, \text{ we label } y \text{ with } k,$$

where τ is a positive number. If there are multiple k's which satisfy this condition, we label y conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

For example, let y be a point contained in an $2(n-1)-1$-dimensional face of $\Delta^{2(n-1)}$ such that $y_i = 0$ for one i among $0, 1, 2, \dots, 2(n-1)$ (the i-th component of its coordinates is 0). With $\tau > 0$, we have $\varphi_i > 0$ or $\varphi_i < \tau$[4]. When $\varphi_i > 0$, from $\sum_{j=0}^{2(n-1)} y_j = 1$, $\sum_{j=0}^{2(n-1)} \varphi_j = 1$ and $y_i = 0$ we have

$$\sum_{j=0, j\neq i}^{2(n-1)} y_j > \sum_{j=0, j\neq i}^{2(n-1)} \varphi_j.$$

Then, for at least one j (denote it by k) we have $y_k > \varphi_k$, and we label y with k, where k is one of the numbers which satisfy $y_k > \varphi_k$. Since $\varphi_i > y_i$, i does

[2] Consider a sequence $(y_m)_{m\geq 1}$ converging to y' and a sequence $(\varphi(y_m))_{m\geq 1}$ such that $\varphi(y_m) \in \Phi(y_m)$ for each m, then closedness of the graph of Φ implies that $(\varphi(y_m)_{m\geq 1}$ converges to a point in $\Phi(y')$. Similarly, consider a sequence $(y'_m)_{m\geq 1}$ converging to y and a sequence $(\varphi(y'_m))_{m\geq 1}$ such that $\varphi(y'_m) \in \Phi(y'_m)$ for each m, then closedness of the graph of Φ implies that $(\varphi(y'_m))_{m\geq 1}$ converges to a point in $\Phi(y)$.

[3] There may exist a case such that for each $\delta > 0$ we can not take a point $\varphi(y)$ for some vertex y so that $|\varphi(y^0) - \varphi(y)| < \varepsilon$ is satisfied. See Note at the end of this proof about such a case.

[4] In constructive mathematics for any real number a we can not prove that $a \geq 0$ or $a < 0$, that $a > 0$ or $a = 0$ or $a < 0$. But for any distinct real numbers a, b and c such that $a > c$ we can prove that $a > b$ or $b > c$.

not satisfy this condition. Assume $\varphi_i < \tau$. $y_i = 0$ implies $\sum_{j=0, j\neq i}^{2(n-1)} y_j = 1$. Since $\sum_{j=0, j\neq i}^{2(n-1)} \varphi_j \leq 1$, we obtain

$$\sum_{j=0, j\neq i}^{2(n-1)} y_j \geq \sum_{j=0, j\neq i}^{2(n-1)} \varphi_j .$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i}^{2(n-1)} (y_j + \tau) > \sum_{j=0, j\neq i}^{2(n-1)} \varphi_j .$$

There is at least one $j (\neq i)$ which satisfies $y_j + \tau > \varphi_j$. Denote it by k, and we label y with k. k is one of the numbers other than i such that $y_k + \tau > \varphi_k$ is satisfied. i itself satisfies this condition ($y_i + \tau > \varphi_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i. We have proved that we can label the vertices contained in an $2(n-1)-1$-dimensional face of $\Delta^{2(n-1)}$ such that $y_i = 0$ for one i among $0, 1, 2, \ldots, 2(n-1)$ with the numbers other than i. By similar procedures we can show that we can label the vertices contained in an $2(n-1)-2$-dimensional face of $\Delta^{2(n-1)}$ such that $y_i = 0$ for two i's among $0, 1, 2, \ldots, 2(n-1)$ with the numbers other than those i's, and so on.

Consider the case where $y_i = y_{i+1} = 0$. By similar procedures we see that when $\varphi_i > 0$ or $\varphi_{i+1} > 0$,

$$\sum_{j=0, j\neq i,i+1}^{2(n-1)} y_j > \sum_{j=0, j\neq i,i+1}^{2(n-1)} \varphi_j .$$

Then, for at least one j (denote it by k) we have $y_k > \varphi_k$, and we label y with k. On the other hand, when $\varphi_i < \tau$ and $\varphi_{i+1} < \tau$, we have

$$\sum_{j=0, j\neq i,i+1}^{2(n-1)} y_j \geq \sum_{j=0, j\neq i,i+1}^{2(n-1)} \varphi_j .$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i,i+1}^{2(n-1)} (y_j + \tau) > \sum_{j=0, j\neq i,i+1}^{2(n-1)} \varphi_j .$$

Thus, there is at least one $j(\neq i, i+1)$ which satisfies $y_j + \tau > \varphi_j$. Denote it by k, and we label y with k.

Next consider the case where $y_i = 0$ for all i other than $2(n-1)$. If for some i $\varphi_i > 0$, then we have $y_{2(n-1)} > \varphi_{2(n-1)}$, and label y with $2(n-1)$. On the other hand, if $\varphi_j < \tau$ for all $j \neq 2(n-1)$, then we obtain $y_{2(n-1)} \geq \varphi_{2(n-1)}$. It implies $y_{2(n-1)} + \tau > \varphi_{2(n-1)}$. Thus, we can label y with $2(n-1)$.

Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K.

3. Let $y^0, y^1, \dots$ and $y^{2(n-1)}$ be the vertices of a fully labeled simplex. We name these vertices so that $y^0, y^1, \dots, y^{2(n-1)}$ are labeled, respectively, with 0, 1, ..., $2(n-1)$. The values of Φ at theses vertices are $\Phi(y^0)$, $\Phi(y^1)$, ... and $\Phi(y^{2(n-1)})$. Take points $\varphi(y^0), \varphi(y^0), \dots \varphi(y^{2(n-1)})$ such that $\varphi(y^0) \in \Phi(y^0), \varphi(y^1) \in \Phi(y^1)$, ... $\varphi(y^{2(n-1)}) \in \Phi(y^{2(n-1)})$. The i-th components of y^0 and $\varphi(y^0)$ are denoted by y_i^0 and $\varphi(y^0)_i$, and so on.

   By our assumption in 2 of this proof when the distance between y^0 and y^1 ($|y^0 - y^1|$) is smaller than δ, the distance between $\varphi(y^0)$ and $\varphi(y^1)$ ($|\varphi(y^0) - \varphi(y^1)|$) is smaller than ε. We can make δ satisfying $\delta < \varepsilon$[5]. Suppose $\tau > 0$. About y^0, from the labeling rules we have $y_0^0 + \tau > \varphi(y^0)_0$. About y^1, also from the labeling rules we have $y_1^1 + \tau > \varphi(y^1)_1$ which implies $y_1^1 > \varphi(y^1)_1 - \tau$. $|y^0 - y^1| < \delta$ implies $|\varphi(y^0) - \varphi(y^1)| < \varepsilon$, which means $\varphi(y^1)_1 > \varphi(y^0)_1 - \varepsilon$. On the other hand, $|y^0 - y^1| < \delta$ means $y_1^0 > y_1^1 - \delta$. Thus, from

$$y_1^0 > y_1^1 - \delta,\ y_1^1 > \varphi(y^1)_1 - \tau,\ \varphi(y^1)_1 > \varphi(y^0)_1 - \varepsilon$$

   we obtain

$$y_1^0 > \varphi(y^0)_1 - \delta - \varepsilon - \tau > \varphi(y^0)_1 - 2\varepsilon - \tau$$

   By similar arguments, for each i other than 0,

$$y_i^0 > \varphi(y^0)_i - 2\varepsilon - \tau. \tag{3.3}$$

   For $i = 0$ we have $y_0^0 + \tau > \varphi(y^0)_0$. Then,

$$y_0^0 > \varphi(y^0)_0 - \tau \tag{3.4}$$

   Adding (3.3) and (3.4) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j\neq k}^{2(n-1)} y_j^0 > \sum_{j=0, j\neq k}^{2(n-1)} \varphi(y^0)_j - 2[2(n-1)-1]\varepsilon - 2(n-1)\tau.$$

[5] For example, for $\delta < 1$ and $\varepsilon < 1$, if when $|y^0 - y^1| < \delta$ we have $|\varphi(y^0) - \varphi(y^1)| < \varepsilon$, then we have $|\varphi(y^0) - \varphi(y^1)| < \varepsilon$ also when $|y^0 - y^1| < \delta\varepsilon < \varepsilon$.

From $\sum_{j=0}^{2(n-1)} y_j^0 = 1$, $\sum_{j=0}^{2(n-1)} \varphi(y^0)_j = 1$ we have $1 - y_k^0 > 1 - \varphi(y^0)_k - 2[2(n-1)-1]\varepsilon - 2(n-1)\tau$, which is rewritten as

$$y_k^0 < \varphi(y^0)_k + 2[2(n-1)-1]\varepsilon + 2(n-1)\tau.$$

Since (3.3) implies $y_k^0 > \varphi(y^0)_k - 2\varepsilon - \tau$, we have

$$\varphi(y^0)_k - 2\varepsilon - \tau < y_k^0 < \varphi(y^0)_k + 2[2(n-1)-1]\varepsilon + 2(n-1)\tau.$$

Thus,

$$|y_k^0 - \varphi(y^0)_k| < 2[2(n-1)-1]\varepsilon + 2(n-1)\tau \tag{3.5}$$

is derived. On the other hand, adding (3.3) from 1 to $2(n-1)$ yields

$$\sum_{j=1}^{2(n-1)} y_j^0 > \sum_{j=1}^{2(n-1)} \varphi(y^0)_j - 4(n-1)\varepsilon - 2(n-1)\tau.$$

From $\sum_{j=0}^{2(n-1)} y_j^0 = 1$, $\sum_{j=0}^{2(n-1)} \varphi(y^0)_j = 1$ we have

$$1 - y_0^0 > 1 - \varphi(y^0)_0 - 4(n-1)\varepsilon - 2(n-1)\tau. \tag{3.6}$$

Then, from (3.4) and (3.6) we get

$$|y_0^0 - \varphi(y^0)_0| < 4(n-1)\varepsilon + 2(n-1)\tau. \tag{3.7}$$

Since n is finite, redefining $4(n-1)\varepsilon + 2(n-1)\tau$ as ε, (3.5) and (3.7) yield

$$|y_i^0 - \varphi(y_i^0)| < \varepsilon \text{ for all } i. \tag{3.8}$$

Appropriately selecting $\varphi(y^j)$ for each j we can show that each y^i satisfies (3.8).

4. Denote one of the points which satisfy (3.8) by $y^* = (x^*, g^*)$. Let

$$\gamma = \sum_{j=0}^{n} \max\left(g_j^* - \frac{1}{n}, 0\right).$$

Then, for $\varepsilon > 0$

$$\left| x_i^* - \frac{x_i^* + \max\left(g_i^* - \frac{1}{n}, 0\right)}{1 + \sum_{j=0}^{n} \max\left(g_j^* - \frac{1}{n}, 0\right)} \right| < \varepsilon.$$

It yields

$$\left|\gamma x_i^* - \max\left(g_i^* - \frac{1}{n}, 0\right)\right| < (1+\gamma)\varepsilon.$$

This means

$$-(1+\gamma)\varepsilon + \gamma x_i^* < \max\left(g_i^* - \frac{1}{n}, 0\right) < (1+\gamma)\varepsilon + \gamma x_i^*. \tag{3.9}$$

Also, appropriately redefining $\varepsilon > 0$ we have

$$|g^* - co(G(x^*))| < \varepsilon.$$

Since γ is finite, if $x_i^* < \varepsilon$ we have

$$\max\left(g_i^* - \frac{1}{n}, 0\right) < \varepsilon.$$

$x_i^* > 0$ for some i because $\sum_{i=0}^{n} x_i^* = 1$. Assume that for one such i

$$\max\left(g_i^* - \frac{1}{n}, 0\right) = g_i^* - \frac{1}{n} > 0.$$

Then, $\gamma > 0$ and from (3.9) we have $\max\left(g_i^* - \frac{1}{n}, 0\right) > 0$ for all i such that $x_i^* > 0$. But, since $\sum_{i=0}^{n} g_i^* = 1$, $\max\left(g_i^* - \frac{1}{n}, 0\right) > 0$ can not hold for all i. Thus, for at least one i $x_i^* < \varepsilon$ and $\max\left(g_i^* - \frac{1}{n}, 0\right) < \varepsilon$.

From the definition of $G(x^*)$ there exists S such that $i \in S$ and $f(x^*) \in V(S)$ for all i such that $x_i^* > 0$. Since $f(x^*) \geq 0$, by the boundedness of $V(S)$ we have $f(x_i^*) < q$ for i such that $x_i^* > 0$. On the other hand, for i such that $x_i^* < \varepsilon$ we have $f(x_i^*) < t\varepsilon$ where t is a finite positive number. Thus, $t\varepsilon$ may also be arbitrarily small. It contradicts 2 of Lemma 3.1. Thus, we have $\max\left(g_i^* - \frac{1}{n}, 0\right) < \varepsilon$ for all i, and so $g_i^* - \frac{1}{n} < \varepsilon$.

If for some i $g_i^* - \frac{1}{n} < 0$, $\sum_{i=1}^{n} g_i^* = 1$ does not hold. Therefore, for all i we have $-\varepsilon < g_i^* - \frac{1}{n} < \varepsilon$, that is, $|g_i^* - \frac{1}{n}| < \varepsilon$. From $|g^* - co(G(x^*))| < \varepsilon$, we obtain

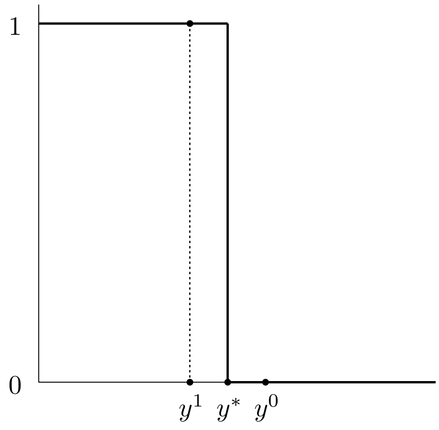

Figure 3: A multi-function in 1-dimensional case

$|m^N - co(G(x^*))| < \varepsilon$. Let $F = \{S \subset N | f(x^*) \in V(S)\}$. Then, by (3.2) F is an approximately balanced collection of sets.

Let $u^* = f(x^*)$. From the definition of G we have $u^* \in \bigcap_{S \in F} V(S)$. Since the game is an approximately balanced game, we have $u^* \in V(N)$. $u^*(= f(x^*)) \geq 0$ and Assumption 1 mean that each component of u^* satisfies $u_i^* < q$. Assume that there exists a vector v such that $u_i^* < v_i < q$ for all $i \in S$. By Assumption 2 and 3 there exists a vector v' such that $u_i^* < v_i' < q$ for all $i \in N$, and it is included in W. But, then u^* can not be included in ∂W. Thus, for all $S \subset N$ we have $u_i^* > v_i - \varepsilon$ for some $i \in S$, and u^* is included in the approximate core.

Note

There may exist a case such that for any $\delta > 0$ we can not take a point $\varphi(y)$ for some vertex y so that $|\varphi(y^0) - \varphi(y)| < \varepsilon$ is satisfied. An example in a 1-dimensional case is a multi-function from [0, 1] to [0, 1] depicted in Figure 3. The coordinates of the points 0 and 1 are, respectively, (0, 1) and (1, 0). And coordinates of other points in [0, 1] are similar. Even if $|y^0 - y^1| < \delta$ for any $\delta < 0$, $|\varphi(y^0) - \varphi(y^1)| > 0$. y^0 and y^1 are, respectively, numbered with 0 and 1. In such a case we must consider further partition of a simplex $[y^0, y^1]$ and take a limit when $\delta \to 0$. At the limit of vertices of a fully labeled simplex y^* there are points $\varphi^1(y^*) \in \Phi(y^*)$ and $\varphi^2(y^*) \in \Phi(y^*)$ such that

$$y_0^* > \varphi^1(y^*)_0 - \tau \text{ and } y_1^* > \varphi^2(y^*)_1 - \tau.$$

Since $y_0^* + y_1^* = 1$ and $\varphi^2(y^*)_0 + \varphi^2(y^*)_1 = 1$, the latter implies

$$y_0^* < \varphi^2(y^*)_0 + \tau.$$

Thus,

$$\varphi^1(y^*)_0 - \tau < y_0^* < \varphi^2(y^*)_0 + \tau.$$

Define a point in $\Phi(y^*)$ by

$$\varphi^*(y^*) = \alpha\varphi^1(y^*) + (1-\alpha)\varphi^2(y^*),\ 0 \le \alpha \le 1.$$

By the convexity of $\Phi(y^*)$, $\varphi^*(y^*) \in \Phi(y^*)$. Let

$$\alpha = \frac{\varphi^2(y^*)_0 + \tau - y_0^*}{[\varphi^2(y^*)_0 + \tau - y_0^*] + [y_0^* - \varphi^1(y^*)_0 + \tau]} = \frac{\varphi^2(y^*)_0 + \tau - y_0^*}{\varphi^2(y^*)_0 - \varphi^1(y^*)_0 + 2\tau},$$

and

$$1 - \alpha = \frac{y_0^* - \varphi^1(y^*)_0 + \tau}{\varphi^2(y^*)_0 - \varphi^1(y^*)_0 + 2\tau}.$$

Then,

$$\varphi^*(y^*)_0 = \frac{\varphi^1(y^*)_0(\tau - y_0^*) + \varphi^2(y^*)_0(\tau + y_0^*)}{\varphi^2(y^*)_0 - \varphi^1(y^*)_0 + 2\tau}.$$

And so we have

$$y_0^* - \varphi^*(y^*)_0 = \frac{\tau[2y_0^* - \varphi^1(y^*)_0 - \varphi^2(y^*)_0]}{\varphi^2(y^*)_0 - \varphi^1(y^*)_0 + 2\tau}.$$

Since τ may be arbitrarily small, for any $\varepsilon > 0$ we obtain

$$|y_0^* - \varphi^*(y^*)_0| < \varepsilon.$$

Similarly

$$|y_1^* - \varphi^*(y^*)_1| < \varepsilon$$

is derived.

A case of higher dimension is similar. ■

4. Concluding Remarks

In this paper we have presented a proof of the existence of an approximate core in an NTU game directly by Sperner's lemma from the viewpoint of constructive mathematics. We are studying some related problems such as the existence of an approximate equilibrium in a competitive economy with multi-valued demand and supply functions, a constructive version of the Fan-Glicksberg fixed point theorem for multi-functions in a locally convex space and its application to a proof of the existence of an approximate Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions.

Appendix

Proof of Sperner's lemma

We prove this lemma by induction about the dimension of Δ. When $n = 0$, we have only one point with the number 0. It is the unique 0-dimensional simplex. Therefore the lemma is trivial. When $n = 1$, a partitioned 1-dimensional simplex is a segmented line. The endpoints of the line are labeled distinctly, with 0 and 1. Hence in moving from endpoint 0 to endpoint 1 the labeling must switch an odd number of times, that is, an odd number of edges labeled with 0 an 1 may be located in this way.

Next consider the case of 2 dimension. Assume that we have partitioned a 2-dimensional simplex (triangle) Δ as explained above. Consider the face of Δ labeled with 0 and 1[6]. It is the base of the triangle in Figure 4. Now we introduce a dual graph that has its nodes in each small triangle of K plus one extra node outside the face of Δ labeled with 0 and 1 (putting a dot in each small triangle, and one dot outside Δ). We define edges of the graph that connect two nodes if they share a side labeled with 0 and 1. See Figure 4. White circles are nodes of the graph, and thick lines are its edges. Since from the result of 1-dimensional case there are an odd number of faces of K labeled with 0 and 1 contained in the face of Δ labeled with 0 and 1, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the triangle which has odd degree. Each node of our graph except for the outside node is contained in one of small triangles of K. Therefore, if a small triangle of K has one face labeled with 0 and 1, the degree of the node in that triangle is 1; if a small triangle of K has two such faces, the degree of the node in that triangle is 2, and if a small triangle of K has no such face, the degree of the node in that triangle is 0. Thus, if the degree of a node is odd, it must be 1, and then the small triangle which contains this node is labeled with 0, 1 and 2 (fully labeled). In Figure 4 triangles which contain one of the nodes A, B, C are fully labeled triangles.

Now assume that the theorem holds for dimensions up to $n-1$. Assume that we have partitioned an n-dimensional simplex Δ. Consider the fully labeled face of Δ which is a fully labeled $n-1$-dimensional simplex. Again we introduce a dual graph that has its nodes in small n-dimensional simplices of K plus one extra node outside the fully labeled face of Δ (putting a dot in each small n-dimensional simplex, and one dot outside Δ). We define the edges of the graph that connect two nodes if they share a face labeled with $0, 1, \dots, n-1$. Since from the result of $n-1$-dimensional case there are an odd number of fully labeled faces of small simplices of K contained in the $n-1$-dimensional fully labeled face of Δ, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since, by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the simplex which has odd degree. Each node of our graph except for the

[6] We call edges of triangle Δ *faces* to distinguish between them and edges of a dual graph which we will consider later.

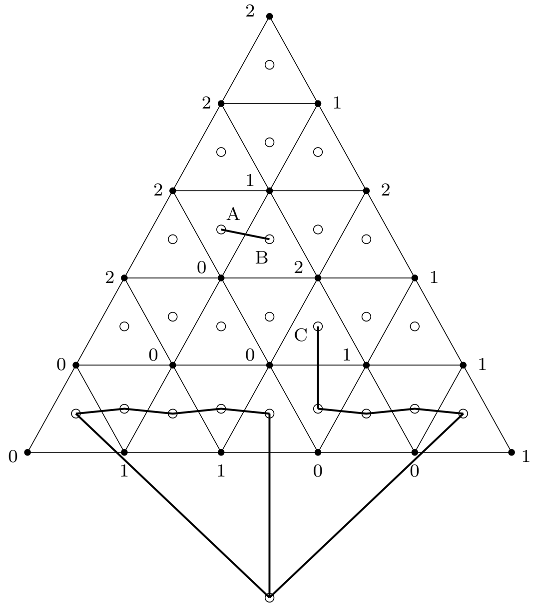

Figure 4: Sperner's lemma

outside node is contained in one of small n-dimensional simplices of K. Therefore, if a small simplex of K has one fully labeled face, the degree of the node in that simplex is 1; if a small simplex of K has two such faces, the degree of the node in that simplex is 2, and if a small simplex of K has no such face, the degree of the node in that simplex is 0. Thus, if the degree of a node is odd, it must be 1, and then the small simplex which contains this node is fully labeled.

> If the number (label) of a vertex other than vertices labeled with 0, 1, ..., $n-1$ of an n-dimensional simplex which contains a fully labeled $n-1$-dimensional face is n, then this n-dimensional simplex has one such face, and this simplex is a fully labeled n-dimensional simplex. On the other hand, if the number of that vertex is other than n, then the n-dimensional simplex has two such faces.

We have completed the proof of Sperner's lemma.

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