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A Proof of the Existence of Approximate Nash Equilibrium in Strategic Game with Multi-Valued Best Responses by Sperner's Lemma: A Constructive Analysis[1]

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Abstract

In this paper we present a constructive proof of the existence of an approximate Nash equilibrium in a finite strategic game with multi-valued best responses using Sperner's lemma. An approximate Nash equilibrium is a state such that mixed strategies chosen by all players are best responses each other within the range of ε. We call such strategies *approximate best responses*. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

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1. Introduction

It is often said that Brouwer's fixed point theorem can not be constructively proved[3]. Therefore, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) and the existence of Nash equilibrium in a finite strategic game with multi-valued best responses also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. See [4] and [10]. In this paper we present a proof of the existence of an approximate Nash equilibrium in a finite strategic game with multi-valued best responses using Sperner's lemma. An approximate Nash equilibrium is a state such that mixed strategies chosen by all players are best responses each other within the range of ε. We call such strategies *approximate best responses*.

Nash himself presented two different proofs of the existence of Nash equilibrium. One is based on Kakutani's fixed point theorem, see [6]. Another is based on Brouwer's fixed point theorem, see [7]. The former is more popular than the latter. This paper is a constructive version of the former. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

2. Approximate Nash equilibrium of strategic game

Let Δ denote an n-dimensional simplex. Let partition the simplex. Figure 1 is an example of partition of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, $\dots$, a point

[3] [5] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics á la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [2] or [4]. Brouwer's fixed point theorem can be constructively, in the sense of constructive mathematics á la Bishop, proved only approximately. The existence of an exact fixed point of a function which satisfies some property of local non-constancy may be constructively proved.

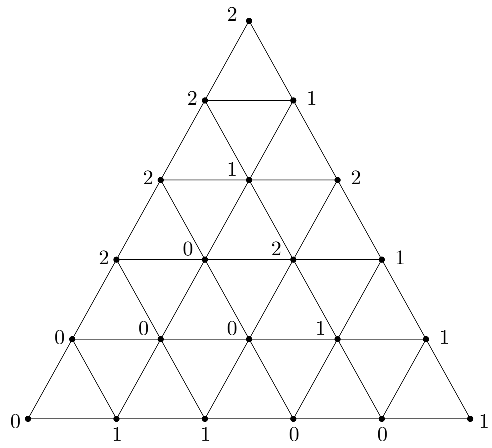

Figure 1: Partition and labeling of 2-dimensional simplex

(0, ..., 0, 1) with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.

2. If a vertex of K is contained in an $n-1$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face.

3. If a vertex of K is contained in an $n-2$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face. And similarly for cases of lower dimension.

4. A vertex contained inside of Δ is labeled with an arbitrary number among 0, 1, $\dots, n$.

A small simplex of K which is labeled with the numbers $0, 1, \dots, n$ is called a *fully labeled simplex*. Sperner's lemma is stated as follows.

Lemma 2.1. [Sperner's lemma] If we label the vertices of K following the rules (1) $\sim$ (4), then there are an odd number of fully labeled simplices, and so there exists at least one fully labeled simplex.

Proof. About constructive proofs of Sperner's lemma see [8] or [9]. ■

Consider a strategic game such that there are n players with m alternative pure strategies for each player. m and n are finite natural numbers larger than 1. Denote the set of strategies for player i by S_i, and denote his each strategy by s_{ij}. His mixed strategy is defined to be a probability distribution over the set of his pure strategies, and is denoted by p_i. Denote the set of all p_i by P_i, which is compact (totally bounded and complete) and convex. The probability that player i chooses a strategy s_{ij} is p_{ij}. Then,

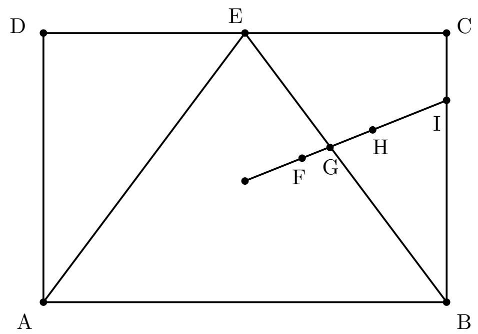

Figure 2: Homeomorphism between $(\Delta^{m-1})^n$ and $\Delta^{n(m-1)}$

$\sum_{j=1}^{m} p_{ij} = 1$ for each i. Call a combination of mixed strategies of all players a *profile*, and denote it by $\mathbf{p}$. $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is a vector which has $n \times m$ components, but only $n(m-1)$ components are independent. The set of all $\mathbf{p}$ is an n-times product of $m-1$-dimensional simplices. It is convex, and homeomorphic to an $n(m-1)$-dimensional simplex. Let denote an n-times product of $m-1$-dimensional simplices by $(\Delta^{m-1})^n$ and an $n(m-1)$-dimensional simplex by $\Delta^{n(m-1)}$. A graph of a multi-function F from $\Delta^{n(m-1)}$ to the set of its inhabited (nonempty) subsets is

$$G(F) = \cup_{\mathbf{p}\in\Delta^{n(m-1)}}\{\mathbf{p}\} \times F(\mathbf{p}).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact.

> Consider sequences $(\mathbf{p}_n)_{n\geq 1}$ and $(\mathbf{q}_n)_{n\geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$. If $\mathbf{p}_n \longrightarrow \mathbf{p}$ and $\mathbf{q}_n \longrightarrow \mathbf{q}$, then $\mathbf{q} \in F(\mathbf{p})$.

According to [3] this means

> If for each neighborhood $U(\mathbf{p}, \varepsilon)$ of $\mathbf{p}$ there exists n_0 such that $\mathbf{p}_n \in U(\mathbf{p}, \varepsilon)$ when $n \geq n_0$, then for the union of neighborhoods $\cup_{\mathbf{q}\in F(\mathbf{p})} V(\mathbf{q}, \varepsilon)$ of points in $F(\mathbf{p})$ there exists n_0' such that $\mathbf{q}_n \in \cup_{\mathbf{q}\in F(\mathbf{p})} V(\mathbf{q}, \varepsilon)$ when $n \geq n_0'$.

Further we consider a uniform version of this property for multi-functions, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. It means that n_0 and n_0' depend on only ε not on $\mathbf{p}$.

A multi-function with uniformly closed graph from $(\Delta^{m-1})^n$ to the set of its inhabited subsets corresponds one to one to a multi-function with uniformly closed graph from $\Delta^{n(m-1)}$ to the set of its inhabited subsets.

The relation between $(\Delta^{m-1})^n$ and $\Delta^{n(m-1)}$ when $n = m = 2$ is illustrated in Figure 2. A point F in this graph corresponds to H, and G corresponds to I.

Denote the expected payoff of player i at a profile $\mathbf{p}$ by $\pi_i(\mathbf{p})$, and his payoff when he chooses a pure strategy s_{ij} at that profile by $\pi_i(s_{ij}, \mathbf{p}_{-i})$. Then, $\pi_i(\mathbf{p})$ is written as follows.

$$\pi_i(\mathbf{p}) = \sum_{\{j: p_{ij}>0\}} p_{ij}\pi_i(s_{ij}, \mathbf{p}_{-i}).$$

$\mathbf{p}_{-i}$ denotes a combination of strategies of players other than i at $\mathbf{p}$. We assume that the payoff of each player is finite. Then, since the expected payoff of each player is linear with respect to probability distributions over the sets of pure strategies of players, it is a uniformly continuous function. We define the set $ABR_i(\mathbf{p}_{-i})$ of *approximate best responses* of player i to $\mathbf{p}_{-i}$ as follows,

$$ABR_i(\mathbf{p}_{-i}) = \{p_i | \pi_i(p_i, \mathbf{p}_{-i}) > \pi_i(p_i', \mathbf{p}_{-i}) - \varepsilon \text{ for all } p_i' \in P_i,\ \varepsilon > 0\}.$$

Each player chooses one of his approximate best responses given a combination of strategies of other players. We call a state where all players choose their approximate best responses each other an *approximate Nash equilibrium*. The set of approximate best responses of all players is represented as follows,

$$\mathbf{ABR}(\mathbf{p}) = (ABR_1(\mathbf{p}_{-1}), ABR_2(\mathbf{p}_{-2}), \dots, ABR_i(\mathbf{p}_{-i}), \dots, ABR_n(\mathbf{p}_{-n})).$$

We show convexity of $\mathbf{ABR}(\mathbf{p})$. It is sufficient that $ABR_i(\mathbf{p}_{-i})$ is convex for each i. Suppose $p_i \in ABR_i(\mathbf{p}_{-i})$ and $p_i'' \in ABR_i(\mathbf{p}_{-i})$. Then,

$$\pi_i(p_i, \mathbf{p}_{-i}) > \pi_i(p_i', \mathbf{p}_{-i}) - \varepsilon \text{ for all } p_i' \in P_i,$$

and

$$\pi_i(p_i'', \mathbf{p}_{-i}) > \pi_i(p_i', \mathbf{p}_{-i}) - \varepsilon \text{ for all } p_i' \in P_i$$

hold. Let $0 \le \lambda \le 1$. Since $\pi_i(\mathbf{p})$ is linear, we have

$$\begin{aligned}\lambda\pi_i(p_i, \mathbf{p}_{-i}) &+ (1-\lambda)\pi_i(p_i'', \mathbf{p}_{-i}) \\ &= \pi_i(\lambda p_i + (1-\lambda)p_i'', \mathbf{p}_{-i}) > \pi_i(p_i', \mathbf{p}_{-i}) - \varepsilon \text{ for all } p_i' \in P_i.\end{aligned}$$

Therefore, $\lambda p_i + (1-\lambda)p_i'' \in ABR_i(\mathbf{p}_{-i})$, and so $ABR_i(\mathbf{p}_{-i})$ is convex.

Next we show that $\mathbf{ABR}(\mathbf{p})$ uniformly has a closed graph. Let $p_i \in ABR_i(\mathbf{p}_{-i})$, p_i'' be a strategy of player i and $\mathbf{p}_{-i}''$ be a combination of strategies of players other than i. By the uniform continuity of $\pi_i(p_i, \mathbf{p}_{-i})$, for any $\varepsilon > 0$ we can select $\delta > 0$ so that when $|(p_i'', \mathbf{p}_{-i}'') - (p_i, \mathbf{p}_{-i})| < \delta$ and $|(p_i', \mathbf{p}_{-i}'') - (p_i', \mathbf{p}_{-i})| < \delta$, then $|\pi_i(p_i'', \mathbf{p}_{-i}'') - \pi_i(p_i, \mathbf{p}_{-i})| < \varepsilon$ and $|\pi_i(p_i', \mathbf{p}_{-i}'') - \pi_i(p_i', \mathbf{p}_{-i})| < \varepsilon$ hold. Since $\pi_i(p_i, \mathbf{p}_{-i}) > \pi_i(p_i', \mathbf{p}_{-i}) - \varepsilon$ for all $p_i' \in P_i$, we have

$$\pi_i(p_i'', \mathbf{p}_{-i}'') > \pi_i(p_i, \mathbf{p}_{-i}) - \varepsilon > \pi_i(p_i', \mathbf{p}_{-i}) - 2\varepsilon > \pi_i(p_i', \mathbf{p}_{-i}'') - 3\varepsilon \text{ for all } p_i' \in P_i.$$

3ε is a positive number which may be arbitrarily small. Thus, redefining 3ε as ε, we have $|p_i'' - ABR_i(\mathbf{p}_{-i}'')| < \varepsilon$ which means $|p_i'' - q_i| < \varepsilon$ for some $q_i \in ABR_i(\mathbf{p}_{-i}'')$.

This relation holds for all i. Therefore, (redefining $n\varepsilon$ as ε) $\mathbf{ABR}(\mathbf{p})$ uniformly has a closed graph.

Now we show the following theorem.

Theorem 2.2. There exists an approximate Nash equilibrium for any finite game.

Proof.

1. Let consider a convex-valued multi-function with uniformly closed graph F from $\Delta^{n(m-1)}$ to the set of its inhabited subsets. We show that we can partition $\Delta^{n(m-1)}$ so that the conditions for Sperner's lemma are satisfied. We partition $\Delta^{n(m-1)}$ according to the method in Sperner's lemma, and label the vertices of simplices constructed by partition of $\Delta^{n(m-1)}$.

   Further suppose that we partition $\Delta^{n(m-1)}$ sufficiently fine so that the distance between any pair of vertices of small simplices constructed by partition is sufficiently small. Let K be the set of small simplices constructed by partition of $\Delta^{n(m-1)}$, and $\mathbf{p} = (p_0, p_1, \dots, p_n)$ and $\mathbf{p}' = (p'_0, p'_1, \dots, p'_n)$ be vertices of a simplex of K. Denote the value of F at $\mathbf{p}$ by $F(\mathbf{p})$, and let $\varphi(\mathbf{p})$ be a point in $F(\mathbf{p})$. Denote the i-th component of $\varphi(\mathbf{p})$ by φ_i, and so on. Since F uniformly has a closed graph, given sufficiently fine partition there exists δ such that if $|\mathbf{p} - \mathbf{p}'| < \delta$, then for $\varepsilon > 0$ $|\varphi(\mathbf{p}) - \varphi(\mathbf{p}')| < \varepsilon$ for any $\varphi(\mathbf{p}) \in F(\mathbf{p})$ and some $\varphi(\mathbf{p}') \in F(\mathbf{p}')$, or for some $\varphi(\mathbf{p}) \in F(\mathbf{p})$ and any $\varphi(\mathbf{p}') \in F(\mathbf{p}')$[4].

   Let $\mathbf{p}^0$ be a vertex of a small $n(m-1)$-dimensional simplex of K which is labeled with 0 by the labelling method which will be explained below. We take a point $\varphi(\mathbf{p}) \in F(\mathbf{p})$ for all other vertices of this simplex so that $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p})| < \varepsilon$ is satisfied[5]

   It is important how to label the vertices contained in the faces of $\Delta^{n(m-1)}$. We label a vertex $\mathbf{p}$ according to the following rule,

   $$\text{If } p_k > \varphi_k \text{ or } p_k + \tau > \varphi_k,\ \text{we label } \mathbf{p} \text{ with } k \text{ for every point } \varphi(\mathbf{p}) \in F(\mathbf{p}),$$

   where τ is a positive number. If there are multiple k's which satisfy this condition, we label $\mathbf{p}$ conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

   For example, let $\mathbf{p}$ be a point contained in an $n(m-1)-1$-dimensional face of $\Delta^{n(m-1)}$ such that $p_i = 0$ for one i among $0, 1, 2, \dots, n(m-1)$ (the i-th

[4]Consider a sequence $(\mathbf{p}(n))_{n\geq 1}$ converging to $\mathbf{p}'$ and a sequence $(\varphi(\mathbf{p})_n)_{n\geq 1}$ such that $\varphi(\mathbf{p}_n)_{n\geq 1} \in F(\mathbf{p}_n)$ for each n, then uniform closedness of the graph of F implies that $(\varphi(\mathbf{p})_n)_{n\geq 1}$ converges to a point in $F(\mathbf{p}')$. Conversely consider a sequence $(\mathbf{p}'_n)_{n\geq 1}$ converging to $\mathbf{p}$ and a sequence $(\varphi(\mathbf{p}')_n)_{n\geq 1}$ such that $\varphi({\mathbf{p}'}_n) \in F(\mathbf{p}'_n)$ for each n, then uniform closedness of the graph of F implies that $\varphi(\mathbf{p}')_n$ converges to a point in $F(\mathbf{p})$.

[5]There may exist a case such that for any $\delta > 0$ we can not take a point $\varphi(\mathbf{p})$ for some vertex $\mathbf{p}$ so that $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p})| < \varepsilon$ is satisfied. See Note at the end of this proof about such a case.

component of its coordinates is 0). With $\tau > 0$, we have $\varphi_i > 0$ or $\varphi_i < \tau$ for each $\varphi(\mathbf{p}) \in F(\mathbf{p})$[6]. When $\varphi_i > 0$, from $\sum_{j=0}^{n(m-1)} p_j = 1$, $\sum_{j=0}^{n(m-1)} \varphi_j = 1$ and $p_i = 0$

$$\sum_{j=0, j\neq i}^{n(m-1)} p_j > \sum_{j=0, j\neq i}^{n(m-1)} \varphi_j .$$

Then, for at least one j (denote it by k) we have $p_k > \varphi_k$, and we label $\mathbf{p}$ with k, where k is one of the numbers which satisfy $p_k > \varphi_k$. Since $\varphi_i > p_i$, i does not satisfy this condition. Assume $\varphi_i < \tau$. $p_i = 0$ implies $\sum_{j=0, j\neq i}^{n(m-1)} p_j = 1$. Since $\sum_{j=0, j\neq i}^{n(m-1)} \varphi_j \leq 1$, we obtain

$$\sum_{j=0, j\neq i}^{n(m-1)} p_j \geq \sum_{j=0, j\neq i}^{n(m-1)} \varphi_j .$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i}^{n(m-1)} (p_j + \tau) > \sum_{j=0, j\neq i}^{n(m-1)} \varphi_j .$$

There is at least one $j (\neq i)$ which satisfies $p_j + \tau > \varphi_j$. Denote it by k, and we label $\mathbf{p}$ with k. k is one of the numbers other than i such that $p_k + \tau > \varphi_k$ is satisfied. i itself satisfies this condition ($p_i + \tau > \varphi_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i. We have proved that we can label the vertices contained in an $n(m-1)-1$-dimensional face of $\Delta^{n(m-1)}$ such that $p_i = 0$ for one i among $0, 1, 2, \ldots, n(m-1)$ with the numbers other than i. By similar procedures we can show that we can label the vertices contained in an $n(m-1)-2$-dimensional face of $\Delta^{n(m-1)}$ such that $p_i = 0$ for two i's among $0, 1, 2, \ldots, n(m-1)$ with the numbers other than those i's, and so on.

Consider the case where $p_i = p_{i+1} = 0$. We see that when $\varphi_i > 0$ or $\varphi_{i+1} > 0$,

$$\sum_{j=0, j\neq i, i+1}^{n(m-1)} p_j > \sum_{j=0, j\neq i, i+1}^{n(m-1)} \varphi_j ,$$

[6]In constructive mathematics for any real number x we can not prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$.

and so for at least one j(denote it by k) we have $p_k > \varphi_k$, and we label $\mathbf{p}$ with k. On the other hand, when $\varphi_i < \tau$ and $\varphi_{i+1} < \tau$, we have

$$\sum_{j=0, j\neq i,i+1}^{n(m-1)} p_j \geq \sum_{j=0, j\neq i,i+1}^{n(m-1)} \varphi_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i,i+1}^{n(m-1)} (p_j + \tau) > \sum_{j=0, j\neq i,i+1}^{n(m-1)} \varphi_j.$$

Thus, there is at least one $j(\neq i, i+1)$ which satisfies $p_j + \tau > \varphi_j$. Denote it by k, and we label $\mathbf{p}$ with k.

Next consider the case where $p_i = 0$ for all i other than $n(m-1)$. If for some i $\varphi_i > 0$, then we have $p_{n(m-1)} > \varphi_{n(m-1)}$, and label $\mathbf{p}$ with $n(m-1)$. On the other hand, if $\varphi_j < \tau$ for all $j \neq n(m-1)$, then we obtain $p_{n(m-1)} \geq \varphi_{n(m-1)}$. It implies $p_{n(m-1)} + \tau > \varphi_{n(m-1)}$. Thus, we can label $\mathbf{p}$ with $n(m-1)$.

Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K.

2. Let $\mathbf{p}^0, \mathbf{p}^1, \ldots$ and $\mathbf{p}^{n(m-1)}$ be the vertices of a fully labeled simplex. We name these vertices so that $\mathbf{p}^0, \mathbf{p}^1, \ldots, \mathbf{p}^{n(m-1)}$ are labeled, respectively, with $0, 1, \ldots, n(m-1)$. The values of F at these vertices are $F(\mathbf{p}^0)$, $F(\mathbf{p}^1), \ldots$ and $F(\mathbf{p}^{n(m-1)})$. Take points $\varphi(\mathbf{p}^0), \varphi(\mathbf{p}^1), \ldots$ and $\varphi(\mathbf{p}^{n(m-1)})$ such that $\varphi(\mathbf{p}^0) \in F(\mathbf{p}^0)$, $\varphi(\mathbf{p}^1) \in F(\mathbf{p}^1), \ldots$ and $\varphi(\mathbf{p}^{n(m-1)}) \in F(\mathbf{p}^{n(m-1)})$. The i-th components of $\mathbf{p}^0$ and $\varphi(\mathbf{p}^0)$ are denoted by $\mathbf{p}^0_i$ and $\varphi(\mathbf{p}^0)_i$, and so on.

By our assumption in (1) of this proof when the distance between $\mathbf{p}^0$ and $\mathbf{p}^1$ ($|\mathbf{p}^0 - \mathbf{p}^1|$) is smaller than δ, the distance between $\varphi(\mathbf{p}^0)$ and $\varphi(\mathbf{p}^1)$ ($|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)|$) is smaller than ε. We can make δ satisfying $\delta < \varepsilon$. Suppose $\tau > 0$. About $\mathbf{p}^0$, from the labeling rules we have $\mathbf{p}^0_0 + \tau > \varphi(\mathbf{p}^0)_0$. About $\mathbf{p}^1$, also from the labeling rules we have $\mathbf{p}^1_1 + \tau > \varphi(\mathbf{p}^1)_1$ which implies $\mathbf{p}^1_1 > \varphi(\mathbf{p}^1)_1 - \tau$. $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)| < \varepsilon$ means $\varphi(\mathbf{p}^1)_1 > \varphi(\mathbf{p}^0)_1 - \varepsilon$. On the other hand, $|\mathbf{p}^0 - \mathbf{p}^1| < \delta$ means $\mathbf{p}^0_1 > \mathbf{p}^1_1 - \delta$. Thus, from

$$\mathbf{p}^0_1 > \mathbf{p}^1_1 - \delta,\ \mathbf{p}^1_1 > \varphi(\mathbf{p}^1)_1 - \tau,\ \varphi(\mathbf{p}^1)_1 > \varphi(\mathbf{p}^0)_1 - \varepsilon$$

we obtain

$$\mathbf{p}^0_1 > \varphi(\mathbf{p}^0)_1 - \delta - \varepsilon - \tau > \varphi(\mathbf{p}^0)_1 - 2\varepsilon - \tau$$

By similar arguments, for each i other than 0,

$$\mathbf{p}^0_i > \varphi(\mathbf{p}^0)_i - 2\varepsilon - \tau. \tag{2.1}$$

For $i = 0$ we have $\mathbf{p}_0^0 + \tau > \varphi(\mathbf{p}^0)_0$. Then,

$$\mathbf{p}_0^0 > \varphi(\mathbf{p}^0)_0 - \tau \tag{2.2}$$

Adding (2.1) and (2.2) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j\neq k}^{n(m-1)} \mathbf{p}_j^0 > \sum_{j=0, j\neq k}^{n(m-1)} \varphi(\mathbf{p}^0)_j - 2[n(m-1)-1]\varepsilon - n(m-1)\tau.$$

From $\sum_{j=0}^{n(m-1)} \mathbf{p}_j^0 = 1$, $\sum_{j=0}^{n(m-1)} \varphi(\mathbf{p}^0)_j = 1$ we have $1 - \mathbf{p}_k^0 > 1 - \varphi(\mathbf{p}^0)_k - 2[n(m-1)-1]\varepsilon - n(m-1)\tau$, which is rewritten as

$$\mathbf{p}_k^0 < \varphi(\mathbf{p}^0)_k + 2[n(m-1)-1]\varepsilon + n(m-1)\tau.$$

Since (2.1) implies $\mathbf{p}_k^0 > \varphi(\mathbf{p}^0)_k - 2\varepsilon - \tau$, we have

$$\varphi(\mathbf{p}^0)_k - 2\varepsilon - \tau < \mathbf{p}_k^0 < \varphi(\mathbf{p}^0)_k + 2[n(m-1)-1]\varepsilon + n(m-1)\tau.$$

Thus,

$$|\mathbf{p}_k^0 - \varphi(\mathbf{p}^0)_k| < 2[n(m-1)-1]\varepsilon + n(m-1)\tau \tag{2.3}$$

is derived. On the other hand, adding (2.1) from 1 to $n(m-1)$ yields

$$\sum_{j=1}^{n(m-1)} \mathbf{p}_j^0 > \sum_{j=1}^{n(m-1)} \varphi(\mathbf{p}^0)_j - 2n(m-1)\varepsilon - n(m-1)\tau.$$

From $\sum_{j=0}^{n(m-1)} \mathbf{p}_j^0 = 1$, $\sum_{j=0}^{n(m-1)} \varphi(\mathbf{p}^0)_j = 1$ we have

$$1 - \mathbf{p}_0^0 > 1 - \varphi(\mathbf{p}^0)_0 - 2n(m-1)\varepsilon - n(m-1)\tau. \tag{2.4}$$

Then, from (2.2) and (2.4) we get

$$|\mathbf{p}_0^0 - \varphi(\mathbf{p}^0)_0| < 2n(m-1)\varepsilon + n(m-1)\tau. \tag{2.5}$$

Since n and m are finite, and ε and τ are positive numbers which may be arbitrarily small, $2n(m-1)\varepsilon + n(m-1)\tau$ and $2[n(m-1)-1]\varepsilon + n(m-1)\tau$ may also be arbitrarily small. Let replace $2n(m-1)\varepsilon + n(m-1)\tau$ by ε, from (2.3) and (2.5) we obtain the following result,

$$|\mathbf{p}_i^0 - \varphi(\mathbf{p}^0)_i| < \varepsilon \text{ for all } i. \tag{2.6}$$

Note that $\varphi(\mathbf{p}^0) \in F(\mathbf{p}^0)$. Appropriately selecting points $\varphi(\mathbf{p}^0)$, $\varphi(\mathbf{p}^1)$, ... and $\varphi(\mathbf{p}^{n(m-1)})$, each vertex of the fully labeled simplex of K satisfies (2.6).

Let denote one of the points that satisfy (2.6) by $\mathbf{p}^* = (p_1^*, p_2^*, \dots, p_n^*)$, and let $F(\mathbf{p})$ be the multi-function of approximate best responses $\mathbf{ABR}(\mathbf{p})$. Then, $|p_i^* - ABR_i(\mathbf{p}_{-i}^*)| < \varepsilon$ for each i. Thus, each p_i^* is a best response within the range of ε, that is, an approximate best response, and so $\mathbf{p}^*$ is an approximate Nash equilibrium.

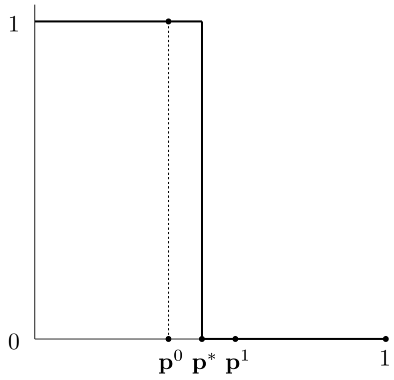

Figure 3: A multi-function in 1-dimensional case

Note

There may exist a case such that for any $\delta > 0$ we can not take a point $\varphi(\mathbf{p})$ for some vertex $\mathbf{p}$ so that $|\varphi(\mathbf{p}) - \varphi(\mathbf{p})| < \varepsilon$ is satisfied. An example in a 1-dimensional case is a multi-function from [0, 1] to [0, 1] depicted in Figure 3. The coordinates of the points 0 and 1 are, respectively, (0, 1) and (1, 0). And coordinates for other points in [0, 1] are similar. Even if $|\mathbf{p}^0 - \mathbf{p}^1| < \delta$ for any $\delta < 0$, $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)| > 0$. $\mathbf{p}^0$ and $\mathbf{p}^1$ are, respectively, numbered with 0 and 1. In such a case we must consider further partition of a simplex $[\mathbf{p}^0, \mathbf{p}^1]$ and take a limit when $\delta \to 0$. At the limit of vertices of a fully labeled simplex $\mathbf{p}^*$ there are points $\varphi^1(\mathbf{p}^*) \in F(\mathbf{p}^*)$ and $\varphi^2(\mathbf{p}^*) \in F(\mathbf{p}^*)$ such that

$$\mathbf{p}_0^* > \varphi^1(\mathbf{p}^*)_0 - \tau \text{ and } \mathbf{p}_1^* > \varphi^2(\mathbf{p}^*)_1 - \tau.$$

Since $\mathbf{p}_0^* + \mathbf{p}_1^* = 1$ and $\varphi^2(\mathbf{p}^*)_0 + \varphi^2(\mathbf{p}^*)_1 = 1$, the latter implies

$$\mathbf{p}_0^* < \varphi^2(\mathbf{p}^*)_0 + \tau.$$

Thus,

$$\varphi^1(\mathbf{p}^*)_0 - \tau < \mathbf{p}_0^* < \varphi^2(\mathbf{p}^*)_0 + \tau.$$

Define a point in $F(\mathbf{p}^*)$ by

$$\varphi^*(\mathbf{p}^*) = \alpha\varphi^1(\mathbf{p}^*) + (1-\alpha)\varphi^2(\mathbf{p}^*),\ 0 \le \alpha \le 1.$$

By the convexity of $F(\mathbf{p}^*)$, $\varphi^*(\mathbf{p}^*) \in F(\mathbf{p}^*)$. Let

$$\alpha = \frac{\varphi^2(\mathbf{p}^*)_0 + \tau - \mathbf{p}_0^*}{[\varphi^2(\mathbf{p}^*)_0 + \tau - \mathbf{p}_0^*] + [\mathbf{p}_0^* - \varphi^1(\mathbf{p}^*)_0 + \tau]} = \frac{\varphi^2(\mathbf{p}^*)_0 + \tau - \mathbf{p}_0^*}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau},$$

and

$$1 - \alpha = \frac{\mathbf{p}_0^* - \varphi^1(\mathbf{p}^*)_0 + \tau}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau}.$$

Then,

$$\varphi^*(\mathbf{p}^*)_0 = \frac{\varphi^1(\mathbf{p}^*)_0(\tau - \mathbf{p}_0^*) + \varphi^2(\mathbf{p}^*)_0(\tau + \mathbf{p}_0^*)}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau}.$$

And so we have

$$\mathbf{p}_0^* - \varphi^*(\mathbf{p}^*)_0 = \frac{\tau[2\mathbf{p}_0^* - \varphi^1(\mathbf{p}^*)_0 - \varphi^2(\mathbf{p}^*)_0]}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau}.$$

Since τ may be arbitrarily small, for any $\varepsilon > 0$ we obtain

$$|\mathbf{p}_0^* - \varphi^*(\mathbf{p}^*)_0| < \varepsilon.$$

Similarly

$$|\mathbf{p}_1^* - \varphi^*(\mathbf{p}^*)_1| < \varepsilon$$

is derived. Therefore, $\mathbf{p}^*$ is an approximate fixed point.

A case of higher dimension is similar. ■

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