# Nash equilibrium of partially asymmetric three-players zero-sum game with two strategic variables

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**Abstract**

We consider a partially asymmetric three-players zero-sum game with two strategic variables. Two players (A and B) have the same payoff functions, and Player C does not. Two strategic variables are $t_i$'s and $s_i$'s for $i = A, B, C$. Mainly we will show the following results.

1. The equilibrium when all players choose $t_i$'s is equivalent to the equilibrium when Players A and B choose $t_i$'s and Player C chooses $s_C$ as their strategic variables.
2. The equilibrium when all players choose $s_i$'s is equivalent to the equilibrium when Players A and B choose $s_i$'s and Player C chooses $t_C$ as their strategic variables.

The equilibrium when all players choose $t_i$'s and the equilibrium when all players choose $s_i$'s are not equivalent although they are equivalent in a symmetric game in which all players have the same payoff functions.

**Keywords:** partially asymmetric three-players zero-sum game, Nash equilibrium, two strategic variables

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# 1 Introduction

We consider a three-players zero-sum game with two strategic variables. Three players are Players A, B and C. Two strategic variables are $t_i$'s and $s_i$'s, $i = A, B, C$. They are related by invertible functions. The game is symmetric for Players A and B in the sense that they have the same payoff functions. On the other hand, Player C may have a different payoff function. Thus, the game is *partially asymmetric*. In Section 3 we will show the following main results.

1. The equilibrium when all players choose $t_i$'s is equivalent to the equilibrium when Players A and B choose $t_i$'s and Player C chooses $s_C$ as their strategic variables.

2. The equilibrium when all players choose $s_i$'s is equivalent to the equilibrium when Players A and B choose $s_i$'s and Player C chooses $t_C$ as their strategic variables.

An example of three-players zero-sum game with two strategic variables is a relative profit maximization game in a three firms oligopoly with differentiated goods. See Section 4. In that section we will show;

1. The equilibrium when all players choose $t_i$'s is not equivalent to the equilibrium when Players A and C choose $t_i$'s and Player B chooses $s_B$ as their strategic variables.

2. The equilibrium when all players choose $t_i$'s is not equivalent to the equilibrium when Players A and B choose $s_i$'s and Player C chooses $t_C$ as their strategic variables.

3. The equilibrium when all players choose $s_i$'s is not equivalent to the equilibrium when Players A and B choose $t_i$'s and Player C chooses $s_C$ as their strategic variables.

4. The equilibrium when all players choose $s_i$'s is not equivalent to the equilibrium when Players A and C choose $s_i$'s and Player B chooses $t_B$ as their strategic variables.

5. The equilibrium when all players choose $t_i$'s is not equivalent to the equilibrium when all players $s_i$'s.

In a symmetric game, in which all players have the same payoff functions, they are all equivalent[1].

In the next section we present a model of this paper and prove a preliminary result which is a variation of Sion's minimax theorem.

# 2 The model

We consider a three-players zero-sum game with two strategic variables. Three players are Players A, B and C. Two strategic variables are $t_i$ and $s_i$, $i = A, B, C$. The game is symmetric for Players A and B in the sense that they have the same payoff functions. On the other hand, Player C may have a different payoff function.

[1]Hattori, Satoh and Tanaka (2018).

$t_i$ is chosen from $T_i$ and $s_i$ is chosen from $S_i$. $T_i$ and $S_i$ are convex and compact sets in linear topological spaces, respectively, for each $i \in \{A, B, C\}$. The relations of the strategic variables are represented by

$$s_i = f_i(t_A, t_B, t_C),\ i = A, B, C,$$

and

$$t_i = g_i(s_A, s_B, s_C),\ i = A, B, C.$$

$(f_A, f_B, f_C)$ and $(g_A, g_B, g_C)$ are continuous, invertible, one-to-one and onto functions. When Players A and B choose $t_A$ and $t_B$ and Player C chooses $s_C$, then $t_C$ is determined according to

$$t_C = g_C(f_A(t_A, t_B, t_C), f_B(t_A, t_B, t_C), s_C).$$

We denote this $t_C$ by $t_C(t_A, t_B, s_C)$.

When Players A and B choose $s_A$ and $s_B$ and Player C chooses $t_C$, then $t_A$ and $t_B$ are determined according to

$$\begin{cases} t_A = g_A(s_A, s_B, f_C(t_A, t_B, t_C)) \\ t_B = g_B(s_A, s_B, f_C(t_A, t_B, t_C)). \end{cases}$$

We denote these $t_A$ and $t_B$ by $t_A(s_A, s_B, t_C)$ and $t_B(s_A, s_B, t_C)$.

When all players choose $s_A$, $s_B$ and $s_C$, $t_A$, then $t_B$ and $t_C$ are determined according to

$$t_A = g_A(s_A, s_B, s_C),\ t_B = g_B(s_A, s_B, s_C),\ t_C = g_C(s_A, s_B, s_C).$$

Denote these $t_A$, $t_B$ and $t_C$ by $t_A(s_A, s_B, s_C)$, $t_B(s_A, s_B, s_C)$ and $t_C(s_A, s_B, s_C)$.

The payoff function of Player $i$ is $u_i,\ i = A, B, C$. It is written as

$$u_i(t_A, t_B, t_C),\ i \in \{A, B, C\}.$$

We assume

> $u_i : T_1 \times T_2 \times T_3 \Rightarrow \mathbb{R}$ for each $i \in \{A, B, C\}$ is continuous on $T_1 \times T_2 \times T_3$. Thus, it is continuous on $S_1 \times S_2 \times S_3$ through $f_i,\ i = A, B, C$. It is quasi-concave on $T_i$ and $S_i$ for a strategy of each other player, and quasi-convex on $T_j,\ j \neq i$ and $S_j,\ j \neq i$ for each $t_i$ and $s_i$.

We do not assume differentiability of the payoff functions.

Symmetry of the game for Players A and B means that in the payoff function of each player, Players A and B are interchangeable. Since the game is a zero-sum game, the sum of the values of the payoff functions of the players is zero.

We assume that all $T_i$'s are identical, and all $S_i$'s are identical. Denote them by $T$ and $S$.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

**Lemma 1.** *Let $X$ and $Y$ be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable, then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of Sion's theorem in Kindler (2005).

Applying this lemma to the situation of this paper, we have the following relations.

$$\max_{t_A\in T}\min_{t_B\in T} u_A(t_A,t_B,t_C)=\min_{t_B\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C),\ \max_{t_B\in T}\min_{t_A\in T} u_B(t_A,t_B,t_C)=\min_{t_A\in T}\max_{t_B\in T} u_B(t_A,t_B,t_C).$$

$$\max_{t_A\in T}\min_{t_C\in T} u_A(t_A,t_B,t_C)=\min_{t_C\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C),\ \max_{t_B\in T}\min_{t_C\in T} u_B(t_A,t_B,t_C)=\min_{t_C\in T}\max_{t_B\in T} u_B(t_A,t_B,t_C).$$

$$\max_{t_A\in T}\min_{t_B\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))=\min_{t_B\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)),$$
$$\max_{t_B\in T}\min_{t_A\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C))=\min_{t_A\in T}\max_{t_B\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C)).$$

$$\max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C))=\min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)),$$
$$\max_{t_B\in T}\min_{s_C\in S} u_B(t_A,t_B,t_C(t_A,t_B,s_C))=\min_{s_C\in S}\max_{t_B\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C)),$$

Further we show the following result.

**Lemma 2.**

$$\min_{t_C\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C)=\min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$$
$$=\max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C))=\max_{t_A\in T}\min_{t_C\in T} u_A(t_A,t_B,t_C),$$

*and*

$$\min_{t_C\in T}\max_{t_B\in T} u_B(t_A,t_B,t_C)=\min_{s_C\in S}\max_{t_B\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C))$$
$$=\max_{t_B\in T}\min_{s_C\in S} u_B(t_A,t_B,t_C(t_A,t_B,s_C))=\max_{t_B\in T}\min_{t_C\in T} u_B(t_A,t_B,t_C).$$

*Proof.* $\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$ is the maximum of $u_A$ with respect to $t_A$ given $t_B$ and $s_C$. Let $\tilde{t}_A(s_C)=\arg\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$, and fix the value of $t_C$ at

$$t_C^0=g_C(f_A(\tilde{t}_A(s_C),t_B,t_C^0),f_B(\tilde{t}_A(s_C),t_B,t_C^0),s_C). \tag{1}$$

We have

$$\max_{t_A\in T} u_A(t_A,t_B,t_C^0)\geq u_A(\tilde{t}_A(s_C),t_B,t_C^0)=\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)),$$

where $\max_{t_A\in T} u_A(t_A,t_B,t_C^0)$ is the maximum of $u_A$ with respect to $t_A$ given the value of $t_C$ at $t_C^0$. We assume that $\tilde{t}_A(s_C)=\arg\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$ is single-valued. By the maximum theorem and continuity of $u_A$, $\tilde{t}_A(s_C)$ is continuous, then any value of $t_C^0$ can be realized by appropriately choosing $s_C$ given $t_B$ according to (1). Therefore,

$$\min_{t_C\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C) \geq \min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)). \tag{2}$$

On the other hand, $\max_{t_A\in T} u_A(t_A,t_B,t_C)$ is the maximum of $u_A$ with respect to $t_A$ given $t_B$ and $t_C$. Let $\tilde{t}_A(t_C) = \arg\max_{t_A\in T} u_A(t_A,t_B,t_C)$, and fix the value of $s_C$ at

$$s_C^0 = f_C(\tilde{t}_A(t_C),t_B,t_C). \tag{3}$$

Thus, we have

$$\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C^0)) \geq u_A(\tilde{t}_A(s_C),t_B,t_C(t_A,t_B,s_C^0)) = \max_{t_A\in T} u_A(t_A,t_B,t_C),$$

where $\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C^0))$ is the maximum of $u_A$ with respect to $t_A$ given the value of $s_C$ at $s_C^0$. We assume that $\tilde{t}_A(t_C) = \arg\max_{t_A\in T} u_A(t_A,t_B,t_C)$ is single-valued. By the maximum theorem and continuity of $u_A$, $\tilde{t}_A(t_C)$ is continuous, then any value of $s_C^0$ can be realized by appropriately choosing $t_C$ given $t_B$ according to (3). Therefore,

$$\min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) \geq \min_{t_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C). \tag{4}$$

Combining (2) and (4), we get

$$\min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{t_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C).$$

Since any value of $s_C$ can be realized by appropriately choosing $t_C$ given $t_A$ and $t_B$, we have

$$\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{t_C\in S} u_A(t_A,t_B,t_C).$$

Thus,

$$\max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) = \max_{t_A\in T}\min_{t_C\in S} u_A(t_A,t_B,t_C).$$

Therefore,

$$\begin{aligned}\min_{t_C\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C) &= \min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)),\\ = \max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) &= \max_{t_A\in T}\min_{t_C\in T} u_A(t_A,t_B,t_C),\end{aligned}$$

given $t_B$.

By similar procedures, we can show

$$\begin{aligned}\min_{t_C\in T}\max_{t_B\in T} u_B(t_A,t_B,t_C) &= \min_{s_C\in S}\max_{t_B\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C)),\\ = \max_{t_B\in T}\min_{s_C\in S} u_B(t_A,t_B,t_C(t_A,t_B,s_C)) &= \max_{t_B\in T}\min_{t_C\in T} u_B(t_A,t_B,t_C),\end{aligned}$$

given $t_A$. □

## 3 The main results

In this section we present the following main result of this paper.

**Theorem 1.** *The equilibrium when all players choose $t_i$'s is equivalent to the equilibrium when Player C chooses $s_C$ and Players A and B choose $t_i$'s as their strategic variables.*

*Proof.* 1. Consider a situation $(t_A, t_B, t_C) = (t, t, t_C)$. By symmetry for Players A and B,

$$\max_{t_A \in T} u_A(t_A, t, t_C) = \max_{t_B \in T} u_B(t, t_B, t_C),$$

and

$$\arg\max_{t_A \in T} u_A(t_A, t, t_C) = \arg\max_{t_B \in T} u_B(t, t_B, t_C) \in T,$$

given $t_C$. Let

$$t_C(t) = \arg\max_{t_C \in T} u_C(t, t, t_C).$$

We assume that it is a single-valued continuous function.

Consider the following function.

$$t \longrightarrow \arg\max_{t_A \in T} u_A(t_A, t, t_C), \text{ given } t_C.$$

This function is continuous and $T$ is compact. Thus, there exists a fixed point given $t_C$. Denote it by $t^*(t_C)$, then

$$t^*(t_C) = \arg\max_{t_A \in T} u_A(t_A, t^*(t_C), t_C) = \arg\max_{t_B \in T} u_B(t^*(t_C), t_B, t_C), \text{ given } t_C.$$

Now we consider the following function.

$$t \longrightarrow t^*(t_C(t)).$$

This also has a fixed point. Denote it by $t^*$ and $t_C(t^*)$ by $t_C^*$, then we have

$$t^* = \arg\max_{t_A \in T} u_A(t_A, t^*, t_C^*) = \arg\max_{t_B \in T} u_B(t^*, t_B, t_C^*),$$

$$t_C^* = \arg\max_{t_C \in T} u_C(t^*, t^*, t_C).$$

$$\max_{t_A \in T} u_A(t_A, t^*, t_C^*) = u_A(t^*, t^*, t_C^*) = \max_{t_B \in T} u_B(t^*, t_B, t_C^*) = u_B(t^*, t^*, t_C^*),$$

and

$$\max_{t_C \in T} u_C(t^*, t^*, t_C) = u_C(t^*, t^*, t_C^*).$$

$(t_A, t_B, t_C) = (t^*, t^*, t_C^*)$ is a Nash equilibrium when all players choose $t_i$'s

2. Because the game is zero-sum,

$$u_A(t^*, t^*, t_C) + u_B(t^*, t^*, t_C) + u_C(t^*, t^*, t_C) = 0.$$

By symmetry $u_A(t^*, t^*, t_C) = u_B(t^*, t^*, t_C)$. Thus,

$$2u_A(t^*, t^*, t_C) + u_C(t^*, t^*, t_C) = 0.$$

This means

$$2u_A(t^*, t^*, t_C) = -u_C(t^*, t^*, t_C),$$

and

$$2 \min_{t_C \in T} u_A(t^*, t^*, t_C) = -\max_{t_C \in T} u_C(t^*, t^*, t_C).$$

From this and symmetry for Players A and B, we get

$$\arg\min_{t_C \in T} u_A(t^*, t^*, t_C) = \arg\min_{t_C \in T} u_B(t^*, t^*, t_C) = \arg\max_{t_C \in T} u_C(t^*, t^*, t_C) = t_C^*.$$

We have

$$\min_{t_C \in T} u_A(t^*, t^*, t_C^*) = u_A(t^*, t^*, t_C^*) = \max_{t_A \in T} u_A(t_A, t^*, t_C^*),$$
$$\min_{t_C \in T} u_B(t^*, t^*, t_C^*) = u_B(t^*, t^*, t_C^*) = \max_{t_B \in T} u_B(t^*, t_B, t_C^*).$$

Therefore,

$$\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t^*, t_C) \le \max_{t_A \in T} u_A(t_A, t^*, t_C^*) = \min_{t_C \in T} u_A(t^*, t^*, t_C) \le \max_{t_A \in T} \min_{t_C \in T} u_A(t_A, t^*, t_C).$$

From Lemma 2 we obtain

$$\begin{aligned} &\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t^*, t_C) = \max_{t_A \in T} u_A(t_A, t^*, t_C^*) = \min_{t_C \in T} u_A(t^*, t^*, t_C) \\ &= \max_{t_A \in T} \min_{t_C \in T} u_A(t_A, t^*, t_C) = \min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) \\ &= \max_{t_A \in T} \min_{s_C \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)). \end{aligned} \tag{5}$$

3. Let

$$s_C^0(t^*) = f_C(t^*, t^*, t_C^*).$$

Since any value of $s_C$ can be realized by appropriately choosing $t_C$,

$$\min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = \min_{t_C \in T} u_A(t^*, t^*, t_C) = u_A(t^*, t^*, t_C^*). \tag{6}$$

Thus,

$$\arg\min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = s_C^0(t^*).$$

(5) and (6) mean

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)). \tag{7}$$

We have

$$\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) \geq u_A(t^*, t^*, t_C(t^*, t^*, s_C)).$$

Therefore,

$$\arg\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \arg\min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = s_C^0(t^*)$$

Thus, by (7)

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \max_{t_A \in T} u_A(t_A, t^*, t_C(t^*, t^*, s_C^0(t^*)))$$
$$= \min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = u_A(t^*, t^*, t_C(t^*, t^*, s_C^0(t^*))).$$

Therefore,

$$\arg\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C^0(t^*))) = t^*. \tag{8}$$

By symmetry for Players A and B,

$$\arg\max_{t_B \in T} u_B(t^*, t_B, t_C(t^*, t_B, s_C^0(t^*))) = t^*. \tag{9}$$

On the other hand, because any value of $s_C$ is realized by appropriately choosing $t_C$,

$$\max_{s_C \in S} u_C(t^*, t^*, t_C(t^*, t^*, s_C)) = \max_{t_C \in T} u_C(t^*, t^*, t_C) = u_C(t^*, t^*, t_C^*).$$

Therefore,

$$\arg\max_{s_C \in S} u_C(t^*, t^*, t_C(t^*, t^*, s_C)) = s_C^0(t^*) = f_C(t^*, t^*, t_C^*). \tag{10}$$

From (8), (9) and (10), $(t^*, t^*, t_C(t^*, t^*, s_C^0(t^*)))$ is a Nash equilibrium which is equivalent to $(t^*, t^*, t_C^*)$.

□

Interchanging $t_i$ and $s_i$ for each player, we can show

**Theorem 2.** *The equilibrium when all players choose $s_i$'s is equivalent to the equilibrium when Player C chooses $t_C$ and Players A and B choose $s_i$'s as their strategic variables.*

# 4 Various examples

Consider a game of relative profit maximization under oligopoly including three firms with differentiated goods[2]. It is a three-players zero-sum game with two strategic variables. The firms are A, B and C. The strategic variables are the outputs and the prices of their goods. We consider the following six patterns of competition.

[2]About relative profit maximization in an oligopoly see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

1. Pattern 1: All firms determine their outputs. It is a Cournot case.

   The inverse demand functions are

   $$p_A = a - x_A - bx_B - bx_C,$$

   $$p_B = a - x_B - bx_A - bx_C,$$

   and

   $$p_C = a - x_C - bx_A - bx_B,$$

   where $0 < b < 1$. $p_A$, $p_B$ and $p_C$ are the prices of the goods of Firms A, B and C, and $x_A$, $x_B$ and $x_C$ are the outputs of them.

2. Pattern 2: Firms A and B determine their outputs, and Firm C determines the price of its good.

   From the inverse demand functions,

   $$p_A = (1-b)a + b^2x_B - bx_B + b^2x_A - x_A + bp_C,$$

   $$p_B = (1-b)a + b^2x_B - x_B + b^2x_A - bx_A + bp_C,$$

   and

   $$x_C = a - bx_B - bx_A - p_C$$

   are derived.

3. Pattern 3: Firms A and C determine their outputs, and Firm B determines the price of its good.

   From the inverse demand functions,

   $$p_A = (1-b)a + b^2x_C - bx_C + b^2x_A - x_A + bp_B,$$

   $$p_C = (1-b)a + b^2x_C - x_C + b^2x_A - bx_A + bp_B,$$

   and

   $$x_B = a - bx_C - bx_A - p_B$$

   are derived.

4. Pattern 4: Firms A and B determine the prices of their goods, and Firm C determines its output.

   From the above inverse demand functions, we obtain

   $$p_C = \frac{(1-b)a + 2b^2x_C - bx_C - x_C + bp_A + bp_B}{1+b},$$

   $$x_B = \frac{(1-b)a + b^2x_C - bx_C + bp_A - p_B}{(1-b)(1+b)},$$

   and

   $$x_A = \frac{(1-b)a + b^2x_C - bx_C - p_A + bp_B}{(1-b)(1+b}.$$

5. Pattern 5: Firms A and C determine the prices of their goods, and Firm B determines its output.

   From the above inverse demand functions, we obtain

$$p_B = \frac{(1-b)a + 2b^2 x_B - bx_B - x_B + bp_C + bp_A}{1+b},$$

$$x_A = \frac{(1-b)a + b^2 x_B - bx_B + bp_C - p_A}{(1-b)(1+b)},$$

   and

$$x_C = \frac{(1-b)a + b^2 x_B - bx_B - p_C + bp_A}{(1-b)(1+b}.$$

6. Pattern 6: All firms determine the prices of their goods. It is a Bertrand case.

   From the inverse demand functions, the direct demand functions are derived as follows;

$$x_A = \frac{(1-b)a - (1+b)p_A + b(p_A + p_C)}{(1-b)(1+2b)},$$

$$x_B = \frac{(1-b)a - (1+b)p_B + b(p_B + p_C)}{(1-b)(1+2b)},$$

   and

$$x_C = \frac{(1-b)a - (1+b)p_C + b(p_A + p_B)}{(1-b)(1+2b)}.$$

The absolute profits of the firms are

$$\pi_A = p_A x_A - c_A x_A,$$

$$\pi_B = p_B x_B - c_B x_B,$$

and

$$\pi_C = p_C x_C - c_C x_C.$$

$c_A$, $c_B$ and $c_C$ are the constant marginal costs of Firms A, B and C. The relative profits of the firms are

$$\psi_A = \pi_A - \frac{\pi_B + \pi_C}{2},$$

$$\psi_B = \pi_B - \frac{\pi_A + \pi_C}{2},$$

and

$$\psi_C = \pi_C - \frac{\pi_A + \pi_B}{2}.$$

The firms determine the values of their strategic variables to maximize the relative profits. We see

$$\psi_A + \psi_B + \psi_C = 0,$$

so the game is zero-sum. We assume $c_A = c_B$, that is, the game is symmetric for Firms A and B. However, $c_C$ is not equal to $c_A$. Thus, the game is partially asymmetric.

We calculate the equilibrium outputs of the firms in the above six patterns.

1. Pattern 1

$$x_A = \frac{bc_C - 4c_A - ab + 4a}{(4-b)(b+2)},$$

$$x_B = \frac{bc_C - 4c_A - ab + 4a}{(4-b)(b+2)},$$

$$x_C = \frac{bc_C + 4c_C - 2bc_A + ab - 4a}{(4-b)(b+2)}.$$

2. Pattern 2

$$x_A = \frac{bc_C - 4c_A - ab + 4a}{(4-b)(b+2)},$$

$$x_B = \frac{bc_C - 4c_A - ab + 4a}{(4-b)(b+2)},$$

$$x_C = \frac{bc_C + 4c_C - 2bc_A + ab - 4a}{(b-4)(b+2)}.$$

3. Pattern 3

$$x_A = \frac{5b^2c_C + 4bc_C - 3b^3c_A + 6b^2c_A + 4bc_A - 16c_A + 3ab^3 - 11ab^2 - 8ab + 16a}{(4-b)(1-b)(b+2)(3b+4)},$$

$$x_B = \frac{bc_C - 4c_A - ab + 4a}{(4-b)(b+2)},$$

$$x_C = \frac{7b^2c_C - 16c_C - 3b^3c_A + 4b^2c_A + 8bc_A + 3ab^3 - 11ab^2 - 8ab + 16a}{(4-b)(1-b)(b+2)(3b+4)}.$$

4. Pattern 4

$$x_A = \frac{2b^2c_C + bc_C + 3b^2c_A - 2bc_A - 4c_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4))},$$

$$x_B = \frac{2b^2c_C + bc_C + 3b^2c_A - 2bc_A - 4c_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4))},$$

$$x_C = \frac{b^2c_C - 3bc_C - 4c_C + 4b^2c_A + 2bc_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4))}.$$

5. Pattern 5

$$x_A = \frac{3b^2c_C - b^3c_C + 4bc_C + 6b^3c_A + 16b^2c_A - 12bc_A - 16c_A - 5ab^3 - 19ab^2 + 8ab + 16a}{(1-b)(b+2)(b+4)(5b+4)},$$

$$x_B = \frac{2b^2c_C + bc_C + 3b^2c_A - 2bc_A - 4c_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4)},$$

$$x_C = \frac{4b^3c_C + 7b^2c_C - 16bc_C - 16c_C + b^3c_A + 12b^2c_A + 8bc_A - 5ab^3 - 19ab^2 + 8ab + 16a}{(1-b)(b+2)(b+4)(5b+4)}.$$

6. Pattern 6

$$x_A = \frac{2b^2c_C + bc_C + 3b^2c_A - 2bc_A - 4c_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4)},$$

$$x_B = \frac{2b^2c_C + bc_C + 3b^2c_A - 2bc_A - 4c_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4)},$$

$$x_C = \frac{b^2c_C - 3bc_C - 4c_C + 4b^2c_A + 2bc_A - 5ab^2 + ab + 4a}{(1-b)(b+2)(5b+4)}.$$

We find that Pattern 1 is equivalent to Pattern 2 (an example of Theorem 1), but it is not equivalent to Pattern 3, and that Pattern 6 is equivalent to Pattern 4 (an example of Theorem 2), but it is not equivalent to Pattern 5. Pattern 1 (Cournot Pattern) and Pattern 6 (Bertrand) are not equivalent unless we have $c_C = c_A$.

# 5 Concluding Remarks

In this paper we have examined equilibria in a partially asymmetric three-players zero-sum game under various situations. We want to extend the results of this paper to a general multi-players zero-sum game.

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