Nash equilibrium in asymmetric multi-players zero-sum game with two strategic variables and only one alien

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Abstract

We consider a partially asymmetric multi-players zero-sum game with two strategic variables. All but one players have the same payoff functions, and one player (Player n) does not. Two strategic variables are t_i's and s_i's for each player i. Mainly we will show the following results. 1) The equilibrium when all players choose t_i's is equivalent to the equilibrium when all but one players choose t_i's and Player n chooses s_n as their strategic variables. 2) The equilibrium when all players choose s_i's is equivalent to the equilibrium when all but one players choose s_i's and Player n chooses t_n as their strategic variables. The equilibrium when all players choose t_i's and the equilibrium when all players choose s_i's are not equivalent although they are equivalent in a symmetric game in which all players have the same payoff functions.

Keywords: partially asymmetric multi-players zero-sum game, Nash equilibrium, two strategic variables

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1 Introduction

We consider a multi-players zero-sum game with two strategic variables. Two strategic variables are t_i and s_i for each player i. They are related by invertible functions. The game is symmetric for all but one player in the sense that they have the same payoff functions. On the other hand, one player (Player n) may have a different payoff function. Thus, the game is partially asymmetric; or there is only one *alien*. In Section 3 we will show the following main results.

1. The equilibrium when all players choose t_i's is equivalent to the equilibrium when all but one players choose t_i's and Player n chooses s_n as their strategic variables.

2. The equilibrium when all players choose s_i's is equivalent to the equilibrium when all but one players choose s_i's and Player n chooses t_n as their strategic variables.

An example of multi-players zero-sum game with two strategic variables is a relative profit maximization game in an oligopoly with differentiated goods. See Section 4. In that section we will show;

1. The equilibrium when all players choose t_i's is not equivalent to the equilibrium when all but one players choose t_i's and one player other than Player n chooses s_i as their strategic variables.

2. The equilibrium when all players choose t_i's is not equivalent to the equilibrium when all but one players choose s_i's and Player n chooses t_n as their strategic variables.

3. The equilibrium when all players choose s_i's is not equivalent to the equilibrium when all but one players choose s_i's and one player other than Player n chooses t_i as their strategic variables.

4. The equilibrium when all players choose s_i's is not equivalent to the equilibrium when all but one players choose t_i's and Player n chooses s_n as their strategic variables.

5. The equilibrium when all players choose t_i's is not equivalent to the equilibrium when all players s_i's.

In these results t_i's are the outputs and s_i's are the prices. In a symmetric game they are all equivalent[1]. In Section 4 we also show that with more than one aliens the equivalence result does not hold.

In the next section we present a model of this paper and prove a preliminary result which is a variation of Sion's minimax theorem.

[1]Hattori, Satoh and Tanaka (2018).

2 The model and the minimax theorem

We consider a multi-players zero-sum game with two strategic variables. There are n players, $n \geq 3$. Two strategic variables are t_i's and s_i's, $i \in \{1, \dots, n\}$. We denote $N = \{1, \dots, n\}$. The game is symmetric for Players $1, 2, \dots, n-1$ in the sense that they have the same payoff functions. On the other hand, Player n may have a different payoff function.

t_i is chosen from T_i and s_i is chosen from S_i. T_i and S_i are convex and compact sets in linear topological spaces, respectively, for each $i \in \{1, \dots, n\}$. The relations of the strategic variables are represented by

$$s_i = f_i(t_1, \dots, t_n),\ i \in N,$$

and

$$t_i = g_i(s_1, \dots, s_n),\ i \in N.$$

$f_i(t_1, \dots, t_n)$ and $g_i(s_1, \dots, s_n)$ are continuous, invertible, one-to-one and onto functions. We assume that all T_i's are identical, and all S_i's are identical. Denote them by T and S.

When only Player n chooses s_n, then t_n is determined according to

$$t_n = g_n(f_1(t_1, \dots, t_n), \dots, f_{n-1}(t_1, \dots, t_n), s_n).$$

We denote this t_n by $t_n(t_1, \dots, t_{n-1}, s_n)$.

When all players choose s_i's, $i \in N$, then t_i's for them are determined according to

$$\begin{cases} t_1 = g_1(s_1, \dots, s_n), \\ \dots \\ t_n = g_n(s_1, \dots, s_n). \end{cases}$$

Denote these t_i's by $t_i(s_1, \dots, s_n)$.

The payoff function of Player i is u_i, $i \in N$. It is written as

$$u_i(t_1, \dots, t_n).$$

We assume

> $u_i : T_1 \times \cdots \times T_n \Rightarrow \mathbb{R}$ for each $i \in N$ is continuous on $T_1 \times \cdots \times T_n$. Thus, it is continuous on $S_1 \times \cdots \times S_n$ through f_i, $i \in N$. It is quasi-concave on T_i and S_i for a strategy of each other player, and quasi-convex on T_j, $j \neq i$ and S_j, $j \neq i$ for each t_i and s_i.

We do not postulate differentiability of the payoff functions.

Symmetry of the game for Players $1, 2, \dots, n-1$ means that these players are interchangeable in the payoff function of each player. Since the game is a zero-sum game, the sum of the values of the payoff functions of the players is zero.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable, then*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

We follow the description of Sion's theorem in Kindler (2005).

Applying this lemma to the situation of this paper such that Player n may choose s_n and the other players choose t_i's as their strategic variables, we have the following relations.

$$\max_{t_i\in T}\min_{t_n\in T} u_i(t_i,t_n,\mathbf{t}_k) = \min_{t_n\in T}\max_{t_i\in T} u_i(t_i,t_n,\mathbf{t}_k).$$

$$\max_{t_i\in T}\min_{s_n\in S} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k) = \min_{s_n\in S}\max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k),$$

where $\mathbf{t}_k$ is a vector of $t_k,\ k \neq i,n$, of the players other than Players i and n who choose t_k's as their strategic variables. $u_i(t_i,t_n,\mathbf{t}_k)$ is the payoff of Player i when Players i and n choose t_i and t_n. On the other hand, $u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k)$ means the payoff of Player i when he chooses t_i and Player n chooses s_n.

We show the following results.

Lemma 2.

$$\begin{aligned}\min_{t_n\in T}\max_{t_i\in T} u_i(t_i,t_n,\mathbf{t}_k) &= \min_{s_n\in S}\max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k)\\ &= \max_{t_i\in T}\min_{s_n\in S} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k) = \max_{t_i\in T}\min_{t_n\in T} u_i(t_i,t_n,\mathbf{t}_k),\end{aligned}$$

Proof. $\max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k)$ is the maximum of u_i with respect to t_i given s_n. Let $\bar{t}_i(s_n) = \arg\max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k)$, and fix the value of t_n at

$$t_n^0 = g_n(f_i(\bar{t}_i(s_n),t_n^0,\mathbf{t}_k),\mathbf{f}_k,s_n), \tag{1}$$

where $\mathbf{f}_k$ denotes a vector of the values of s_k's of players who choose t_k's. We have

$$\max_{t_i\in T} u_i(t_i,t_n^0,\mathbf{t}_k) \geq u_i(\bar{t}_i(s_n),t_n^0,\mathbf{t}_k) = \max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k),$$

where $\max_{t_i\in T} u_i(t_i,t_n^0,\mathbf{t}_k)$ is the maximum of u_i with respect to t_i given the value of t_n at t_n^0. We assume that $\bar{t}_i(s_n) = \arg\max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k)$ is single-valued. By the maximum theorem and continuity of u_i, $\bar{t}_i(s_n)$ is continuous, then any value of t_n^0 can be realized by appropriately choosing s_n according to (1). Therefore,

$$\min_{t_n\in T}\max_{t_i\in T} u_i(t_i,t_n,\mathbf{t}_k) \geq \min_{s_n\in S}\max_{t_i\in T} u_i(t_i,t_n(t_i,s_n,\mathbf{t}_k),\mathbf{t}_k). \tag{2}$$

On the other hand, $\max_{t_i\in T} u_i(t_i,t_n,\mathbf{t}_k)$ is the maximum of u_i with respect to t_i given t_n. Let $\bar{t}_i(t_n) = \arg\max_{t_i\in T} u_i(t_i,t_n,\mathbf{t}_k)$, and fix the value of s_n at

$$s_n^0 = f_n(\bar{t}_i(t_n), t_n, \mathbf{t}_k). \tag{3}$$

We have

$$\max_{t_i\in T} u_i(t_i, t_n(t_i, s_n^0, \mathbf{t}_k), \mathbf{t}_k) \geq u_i(\bar{t}_i(s_n), t_n(t_i, s_n^0, \mathbf{t}_k), \mathbf{t}_k) = \max_{t_i\in T} u_i(t_i, t_n, \mathbf{t}_k),$$

where $\max_{t_i\in T} u_i(t_i, t_n(t_i, s_n^0, \mathbf{t}_k), \mathbf{t}_k)$ is the maximum of u_i with respect to t_i given the value of s_n at s_n^0. We assume that $\bar{t}_i(t_n) = \arg\max_{t_i\in T} u_i(t_i,t_n,\mathbf{t}_k)$ is single-valued. By the maximum theorem and continuity of u_i, $\bar{t}_i(t_n)$ is continuous, then any value of s_n^0 can be realized by appropriately choosing t_n according to (3). Therefore,

$$\min_{s_n\in S}\max_{t_i\in T} u_i(t_i, t_n(t_i, s_n, \mathbf{t}_k), \mathbf{t}_k) \geq \min_{t_n\in T}\max_{t_i\in T} u_i(t_i, t_n, \mathbf{t}_k). \tag{4}$$

Combining (2) and (4), we get

$$\min_{s_n\in S}\max_{t_i\in T} u_i(t_i, t_n(t_i, s_n, \mathbf{t}_k), \mathbf{t}_k) = \min_{t_n\in T}\max_{t_i\in T} u_i(t_i, t_n, \mathbf{t}_k).$$

Since any value of s_n can be realized by appropriately choosing t_n, we have

$$\min_{s_n\in S} u_i(t_i, t_n(t_i, s_n, \mathbf{t}_k), \mathbf{t}_k) = \min_{t_n\in T} u_i(t_i, t_n, \mathbf{t}_k).$$

Thus,

$$\max_{t_i\in T}\min_{s_n\in S} u_i(t_i, t_n(t_i, s_n, \mathbf{t}_k), \mathbf{t}_k) = \max_{t_i\in T}\min_{t_n\in T} u_i(t_i, t_n, \mathbf{t}_k).$$

Therefore,

$$\begin{aligned}\min_{t_n\in T}\max_{t_i\in T} u_i(t_i, t_n, \mathbf{t}_k) &= \min_{s_n\in S}\max_{t_i\in T} u_i(t_i, t_n(t_i, s_n, \mathbf{t}_k), \mathbf{t}_k),\\ = \max_{t_i\in T}\min_{s_n\in S} u_i(t_i, t_n(t_i, s_n, \mathbf{t}_k), \mathbf{t}_k) &= \max_{t_i\in T}\min_{t_n\in T} u_i(t_i, t_n, \mathbf{t}_k).\end{aligned}$$

□

3 The main results

In this section we present the main result of this paper.

Theorem 1. *The equilibrium where all players choose t_i's is equivalent to the equilibrium when one player (Player n) chooses s_n and all other players choose t_i's as their strategic variables.*

Proof. 1. Consider a situation $(t_1, \dots, t_{n-1}, t_n) = (t, \dots, t, t_n)$. By symmetry for Players 1, 2, ..., $n-1$

$$\max_{t_i\in T} u_i(t, \dots, t_i, \dots, t, t_n) = \max_{t_j\in T} u_j(t, \dots, t_j, \dots, t, t_n), \text{ for any } i, j \neq n,$$

and

$$\arg\max_{t_i\in T} u_i(t,\dots,t_i,\dots,t,t_n) = \arg\max_{t_j\in T} u_j(t,\dots,t_j,\dots,t,t_n) \in T, \text{ for any } i,j \neq n,$$

given t_n.

Let

$$t_n(t) = \arg\max_{t_n\in T} u_n(t,\dots,t,t_n).$$

We assume that it is a single-valued continuous function.

Consider the following function.

$$t \longrightarrow \arg\max_{t_i\in T} u_i(t,\dots,t_i,\dots,t,t_n), \text{ for any } i \neq n, \text{ given } t_n.$$

This function is continuous and T is compact. Thus, there exists a fixed point given t_n. Denote it by $t^*(t_n)$, then

$$t^*(t_n) = \arg\max_{t_i\in T} u_i(t^*(t_n),\dots,t_i,\dots,t^*(t_n),t_n), \text{ for any } i \neq n, \text{ given } t_n.$$

Now we consider the following function.

$$t \longrightarrow t^*(t_n(t)).$$

This also has a fixed point. Denote it by t^* and $t_n(t^*)$ by t_n^*, then we have

$$t^* = \arg\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n^*), \text{ for any } i \neq n,$$

$$t_n^* = \arg\max_{t_n\in T} u_n(t^*,\dots,t^*,t_n),$$

$$\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n^*) = u_i(t^*,\dots,t^*,t_n^*), \text{ for any } i \neq n,$$

and

$$\max_{t_n\in T} u_n(t^*,\dots,t^*,t_n) = u_n(t^*,\dots,t^*,t_n^*).$$

$(t_1,\dots,t_{n-1},t_n) = (t^*,\dots,t^*,t_n^*)$ is a Nash equilibrium when all players choose t_i's as their strategic variables.

2. Because the game is zero-sum,

$$u_1(t^*,\dots,t^*,t_n) + \cdots + u_{n-1}(t^*,\dots,t^*,t_n) + u_n(t^*,\dots,t^*,t_n) = 0.$$

By symmetry for Players 1, 2, ..., $n-1$,

$$(n-1)u_i(t^*,\dots,t^*,t_n) + u_n(t^*,\dots,t^*,t_n) = 0.$$

This means

$$(n-1)u_i(t^*,\dots,t^*,t_n) = -u_n(t^*,\dots,t^*,t_n),$$

and

$$(n-1)\min_{t_n\in T} u_i(t^*,\dots,t^*,t_n) = -\max_{t_n\in T} u_n(t^*,\dots,t^*,t_n).$$

From this we get

$$\arg\min_{t_n\in T} u_i(t^*,\dots,t^*,t_n) = \arg\max_{t_n\in T} u_n(t^*,\dots,t^*,t_n) = t_n^*, \text{ for any } i\neq n.$$

We have

$$\min_{t_n\in T} u_i(t^*,\dots,t^*,t_n) = u_i(t^*,\dots,t^*,t_n^*) = \max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n^*), \text{ for any } i\neq n.$$

Thus,

$$\begin{aligned}&\min_{t_n\in T}\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n) \le \max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n^*) = \min_{t_n\in T} u_i(t^*,\dots,t^*,t_n)\\ &\le \max_{t_i\in T}\min_{t_n\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n), \text{ for any } i\neq n.\end{aligned}$$

From Lemma 2 we obtain

$$\begin{aligned}&\min_{t_n\in T}\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n) = \max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n^*)\\ &= \min_{t_n\in T} u_i(t^*,\dots,t^*,t_n) = \max_{t_i\in T}\min_{t_n\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n)\\ &= \min_{s_n\in S}\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n))\\ &= \max_{t_i\in T}\min_{s_n\in S} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n)), \text{ for any } i\neq n.\end{aligned} \tag{5}$$

3. Let

$$s_n^0(t^*) = f_n(t^*,\dots,t^*,t_n^*).$$

Since any value of s_n can be realized by appropriately choosing t_n,

$$\min_{s_n\in S} u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n)) = \min_{t_n\in T} u_i(t^*,\dots,t^*,t_n) = u_i(t^*,\dots,t^*,t_n^*). \tag{6}$$

Thus,

$$\arg\min_{s_n\in S} u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n)) = s_n^0(t^*).$$

(5) and (6) mean

$$\begin{aligned}&\min_{s_n\in S}\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n))\\ &= \min_{s_n\in S} u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n)).\end{aligned} \tag{7}$$

We have

$$\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n)) \ge u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n)).$$

Thus,

$$\arg\min_{s_n\in S}\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n))$$
$$=\arg\min_{s_n\in S} u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n))=s_n^0(t^*).$$

By (7)

$$\min_{s_n\in S}\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n))$$
$$=\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n^0(t^*)))$$
$$=\min_{s_n\in S} u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n))=u_i(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n^0(t^*))).$$

Therefore,

$$\arg\max_{t_i\in T} u_i(t^*,\dots,t_i,\dots,t^*,t_n(t^*,\dots,t_i,\dots,t^*,s_n^0(t^*)))=t^*. \quad (8)$$

This holds for any player $i\neq n$.

On the other hand, because any value of s_n is realized by appropriately choosing t_n,

$$\max_{s_n\in S} u_n(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n))=\max_{t_n\in T} u_n(t^*,\dots,t^*,t_n)=u_n(t^*,\dots,t^*,t_n^*).$$

Therefore,

$$\arg\max_{s_n\in S} u_n(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n))=s_n^0(t^*)=f_C(t^*,\dots,t^*,t_n^*). \quad (9)$$

From (8) and (9), $(t^*,\dots,t^*,t_n(t^*,\dots,t^*,s_n^0(t^*)))$ is a Nash equilibrium which is equivalent to $(t^*,\dots,t^*,t_n^*)$.

□

Interchanging t_i and s_i for each player, we can show

Theorem 2. *The equilibrium where all players choose s_i's is equivalent to the equilibrium when one player (Player n) chooses t_n and all other players choose s_i's as their strategic variables.*

4 Various examples

Consider a game of relative profit maximization under oligopoly including four firms with differentiated goods[2]. It is a four-players zero-sum game with two strategic variables. The firms are A, B, C and D. The strategic variables are the outputs and the prices of their goods. We consider the following four patterns of competition.

[2]About relative profit maximization in an oligopoly see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

1. Pattern 1: All firms determine their outputs. It is a Cournot case.

   The inverse demand functions are

$$p_A = a - x_A - bx_B - bx_C - bx_D,$$

$$p_B = a - x_B - bx_A - bx_C - bx_D,$$

$$p_C = a - x_C - bx_A - bx_B - bx_D,$$

   and

$$p_D = a - x_D - bx_A - bx_B - bx_C,$$

   where $0 < b < 1$. p_A, p_B, p_C and p_D are the prices of the goods of Firms A, B, C and D, and x_A, x_B, x_C and x_D are their outputs.

2. Pattern 2: Firms A, B and C determine their outputs, and Firm D determines the price of its good.

   From the inverse demand functions,

$$p_A = (1-b)a + b^2 x_C - bx_C + b^2 x_B - bx_B + b^2 x_A - x_A + bp_D,$$

$$p_B = (1-b)a + b^2 x_C - bx_C + b^2 x_B - x_B + b^2 x_A - bx_A + bp_D,$$

$$p_C = (1-b)a + b^2 x_C - x_C + b^2 x_B - bx_B + b^2 x_A - bx_A + bp_D,$$

$$x_D = a - bx_C - bx_B - bx_A - p_D$$

   are derived.

3. Pattern 3: Firms A, B and C determine the prices, and Firm D determines the output.

   From the inverse demand functions,

$$x_A = \frac{(1-b)a + b^2 x_D - bx_D + bp_C + bp_B - bp_A - p_A}{(1-b)(2b+1)},$$

$$x_B = \frac{(1-b)a + b^2 x_D - bx_D + bp_C - bp_B - p_B + bp_A}{(1-b)(2b+1)},$$

$$x_C = \frac{(1-b)a + b^2 x_D - bx_D - bp_C - p_C + bp_B + bp_A}{(1-b)(2b+1)},$$

$$p_D = \frac{(1-b)a + 3b^2 x_D - 2bx_D - x_D + bp_C + bp_B + bp_A}{2b+1}.$$

4. Pattern 4: All firms determine the prices. It is a Bertrand case.

   From the inverse demand functions, the direct demand functions are derived as follows;

$$x_A = \frac{(1-b)a + bp_D + bp_C + bp_B - 2bp_A - p_A}{(1-b)(3b+1)},$$

$$x_B = \frac{(1-b)a + bp_D + bp_C - 2bp_B - p_B + bp_A}{(1-b)(3b+1)},$$
$$x_C = \frac{(1-b)a + bp_D - 2bp_C - p_C + bp_B + bp_A}{(1-b)(3b+1)},$$
$$x_D = \frac{(1-b)a - 2bp_D - p_D + bp_C + bp_B + bp_A}{(1-b)(3b+1)}.$$

The absolute profits of the firms are

$$\pi_A = p_A x_A - c_A x_A,$$

$$\pi_B = p_B x_B - c_B x_B,$$

$$\pi_C = p_C x_C - c_C x_C,$$

and

$$\pi_D = p_D x_D - c_D x_D.$$

c_A, c_B, c_C and c_D are the constant marginal costs of Firms A, B, C and D. The relative profits of the firms are

$$\varphi_A = \pi_A - \frac{\pi_B + \pi_C + \pi_D}{3},$$
$$\varphi_B = \pi_B - \frac{\pi_A + \pi_C + \pi_D}{3},$$
$$\varphi_C = \pi_C - \frac{\pi_A + \pi_B + \pi_D}{3},$$

and

$$\varphi_D = \pi_D - \frac{\pi_A + \pi_B + \pi_C}{3}.$$

The firms determine the values of their strategic variables to maximize the relative profits. We see

$$\varphi_A + \varphi_B + \varphi_C + \varphi_D = 0,$$

so the game is zero-sum. We assume $c_A = c_B = c_C$, that is, the game is symmetric for Firms A, B and C. However, c_D is not equal to c_A. Thus, the game is partially asymmetric. Firm D is an alien.

We calculate the equilibrium outputs of the firms in the above four patterns.

1. Pattern 1

$$x_A = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)},$$
$$x_B = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)},$$
$$x_C = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)},$$

and

$$x_D = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)}.$$

2. Pattern 2

$$x_A = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)},$$

$$x_B = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)},$$

$$x_C = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)},$$

and

$$x_D = \frac{bc_D - 3c_A - ab + 3a}{2(3-b)(b+1)}.$$

3. Pattern 3

$$x_A = \frac{3b^2c_D + bc_D + 4b^2c_A - 5bc_A - 3c_A - 7ab^2 + 4ab + 3a}{2(1-b)(b+1)(7b+3)},$$

$$x_B = \frac{3b^2c_D + bc_D + 4b^2c_A - 5bc_A - 3c_A - 7ab^2 + 4ab + 3a}{2(1-b)(b+1)(7b+3)},$$

$$x_C = \frac{3b^2c_D + bc_D + 4b^2c_A - 5bc_A - 3c_A - 7ab^2 + 4ab + 3a}{2(1-b)(b+1)(7b+3)},$$

$$x_D = \frac{3a - 2b^2c_D - 7bc_D - 3c_D + 9b^2c_A + 3bc_A - 7ab^2 + 4ab}{2(1-b)(b+1)(7b+3)}.$$

4. Pattern 4

$$x_A = \frac{3b^2c_D + bc_D + 4b^2c_A - 5bc_A - 3c_A - 7ab^2 + 4ab + 3a}{2(1-b)(b+1)(7b+3)},$$

$$x_B = \frac{3b^2c_D + bc_D + 4b^2c_A - 5bc_A - 3c_A - 7ab^2 + 4ab + 3a}{2(1-b)(b+1)(7b+3)},$$

$$x_C = \frac{3b^2c_D + bc_D + 4b^2c_A - 5bc_A - 3c_A - 7ab^2 + 4ab + 3a}{2(1-b)(b+1)(7b+3)},$$

$$x_D = \frac{3a - 2b^2c_D - 7bc_D - 3c_D + 9b^2c_A + 3bc_A - 7ab^2 + 4ab}{2(1-b)(b+1)(7b+3)}.$$

We find that Pattern 1 is equivalent to Pattern 2 (an example of Theorem 1), but it is not equivalent to Patterns 3 and 4, and that Pattern 4 is equivalent to Pattern 3 (an example of Theorem 2), but it is not equivalent to Patterns 1 and 2.

Next let us examine a case where $c_B = c_A$ and $c_C = c_D$ but $c_A \neq c_D$. Consider the following two patterns of competition.

1. Pattern 1: All firms determine their outputs. It is a Cournot case.

2. Pattern 2: Firms A and B determine their outputs, and Firms C and D determine the prices of their goods, then

$$p_A = \frac{b^2x_B - bx_B + 2b^2x_A - bx_A - x_A + bp_D + bp_C - ab + a}{b+1},$$
$$p_B = \frac{2b^2x_B - bx_B - x_B + b^2x_A - bx_A + bp_D + bp_C - ab + a}{b+1},$$
$$x_C = \frac{b^2x_B - bx_B + b^2x_A - bx_A + bp_D - p_C - ab + a}{(1-b)(b+1)},$$
$$x_D = \frac{b^2x_B - bx_B + b^2x_A - bx_A - p_D + bp_C - ab + a}{(1-b)(b+1)}.$$

We calculate the equilibrium outputs of the firms in these two patterns.

1. Pattern 1

$$x_A = \frac{2bc_D - bc_A - 3c_A - ab + 3a}{2(3-b)(b+1)},$$
$$x_B = \frac{2bc_D - bc_A - 3c_A - ab + 3a}{2(3-b)(b+1)},$$
$$x_C = \frac{3a - bc_D - 3c_D + 2bc_A - ab}{2(3-b)(b+1)},$$
$$x_D = \frac{3a - bc_D - 3c_D + 2bc_A - ab}{2(3-b)(b+1)}.$$

2. Pattern 2

$$x_A = \frac{2bc_D + bc_A - 3c_A - 3ab + 3a}{6(1-b)(b+1)},$$
$$x_B = \frac{2bc_D + bc_A - 3c_A - 3ab + 3a}{6(1-b)(b+1)},$$
$$x_C = \frac{bc_D - 3c_D + 2bc_A - 3ab + 3a}{6(1-b)(b+1)},$$
$$x_D = \frac{bc_D - 3c_D + 2bc_A - 3ab + 3a}{6(1-b)(b+1)}.$$

Patterns 1 and 2 are not equivalent. Therefore, with more than one aliens the equivalence result does not hold.

5 Concluding Remarks

In this paper we have examined equilibria in a partially asymmetric multi-players zero-sum game. We have shown that in an asymmetric zero-sum game with only one alien (a player who has a different payoff function) we get the equivalence result about the choice of strategic variables.

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