Convexity, concavity, super-additivity, and sub-additivity of cost function

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Abstract

With zero fixed cost, convexity of a cost function implies super-additivity, and concavity of a cost function implies sub-additivity. But converse relations do not hold. However, if we assume that the cost function is convex throughout the domain or concave throughout the domain, super-additivity implies convexity and sub-additivity implies concavity, that is, super-additivity and convexity are equivalent, and sub-additivity and concavity are equivalent.

Keywords: convexity, concavity, super-additivity, sub-additivity, cost function

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1 Introduction

Convexity and concavity are important properties for cost functions of firms. Also super-additivity and sub-additivity are other important properties for them. A cost function $c(x)$ is convex when it satisfies

$$c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y) \text{ for } 0 \leq \lambda \leq 1,\ x \geq 0,\ y \geq 0.$$

It is concave when it satisfies

$$c(\lambda x + (1-\lambda)y) \geq \lambda c(x) + (1-\lambda)c(y) \text{ for } 0 \leq \lambda \leq 1,\ x \geq 0,\ y \geq 0.$$

It is super-additive if it satisfies

$$c(x+y) \geq c(x) + c(y),\ \text{for } x \geq 0,\ y \geq 0.$$

It is sub-additive if it satisfies

$$c(x+y) \leq c(x) + c(y),\ \text{for } x \geq 0,\ y \geq 0.$$

It is well known that with zero fixed cost, that is, $c(0) = 0$, convexity implies super-additivity, and concavity implies sub-additivity. But converse relations do not hold. See Bruckner and Ostrow (1962) and Sen and Stamatopoulos (2016). Referring to Bourin and Hiai (2015), Sen and Stamatopoulos (2016) pointed out that the following function is super-additive but it is not convex.

$$xe^{-\frac{1}{x^2}},\ x \geq 0.$$

However, in addition to the zero fixed cost condition if we assume that the cost function is convex throughout the domain or concave throughout the domain, we can show that super-additivity implies convexity, and sub-additivity implies concavity. Then, super-additivity and convexity are equivalent, and sub-additivity and concavity are equivalent.

2 Main results

We show the following theorem.

Theorem 1. (1) *i) If there is no fixed cost, convexity of the cost function implies its super-additivity.*

ii) If there is no fixed cost, concavity of the cost function implies its sub-additivity.

(2) *Suppose that the cost function is convex throughout the domain or concave throughout the domain, and there is no fixed cost.*

i) Super-additivity of the cost function implies convexity.

ii) Sub-additivity of the cost function implies concavity.

Proof. (1) i) Convexity ⇒ super-additivity:

For $0 \le \lambda \le 1$ and $x \ge 0$ convexity implies

$$c(\lambda x + (1-\lambda)\cdot 0) \le \lambda c(x) + (1-\lambda)c(0) = \lambda c(x).$$

Thus,

$$c(\lambda x) \le \lambda c(x).$$

Then, for $x > 0$ and $y \ge 0$

$$\begin{aligned} c(x) + c(y) =& c\left(\frac{x}{x+y}(x+y)\right) + c\left(\frac{y}{x+y}(x+y)\right) \\ \le& \frac{x}{x+y}c(x+y) + \frac{y}{x+y}c(x+y) = c(x+y). \end{aligned}$$

ii) Concavity ⇒ sub-additivity:

For $0 \le \lambda \le 1$ and $x \ge 0$ concavity implies

$$c(\lambda x + (1-\lambda)\cdot 0) \ge \lambda c(x) + (1-\lambda)c(0) = \lambda c(x).$$

Thus,

$$c(\lambda x) \ge \lambda c(x).$$

Then, for $x > 0$ and $y \ge 0$

$$\begin{aligned} c(x) + c(y) =& c\left(\frac{x}{x+y}(x+y)\right) + c\left(\frac{y}{x+y}(x+y)\right) \\ \ge& \frac{x}{x+y}c(x+y) + \frac{y}{x+y}c(x+y) = c(x+y). \end{aligned}$$

(2) i) Super-additivity ⇒ convexity:

Let $x \ge 0$ and $0 \le \lambda \le 1$. By super-additivity,

$$\begin{aligned} c(x) =& c(\lambda x + (1-\lambda)x) \ge c(\lambda x) + c((1-\lambda)x) \qquad (1) \\ =& c(\lambda x + (1-\lambda)\cdot 0) + c((1-\lambda)x + \lambda \cdot 0). \end{aligned}$$

By the zero fixed cost condition,

$$c(x) = \lambda c(x) + (1-\lambda)c(0) + (1-\lambda)c(x) + \lambda c(0).$$

Then, (1) means that

$$\lambda c(x) + (1-\lambda)c(0) \ge c(\lambda x + (1-\lambda)\cdot 0). \qquad (2)$$

or

$$(1-\lambda)c(x) + \lambda c(0) \ge c((1-\lambda)x + \lambda \cdot 0). \qquad (3)$$

holds. By the assumption that $c(x)$ is convex throughout the domain or concave throughout the domain, if either of (2) and (3) holds, $c(x)$ is convex.

Since λ is arbitrary, this result implies that super-additivity of $c(x)$ means its convexity in an interval including the origin.

ii) Sub-additivity ⇒ concavity:

Let $x \geq 0$ and $0 \leq \lambda \leq 1$. By sub-additivity,

$$\begin{aligned} c(x) =& c(\lambda x + (1-\lambda)x) \leq c(\lambda x) + c((1-\lambda)x) \\ =& c(\lambda x + (1-\lambda)\cdot 0) + c((1-\lambda)x + \lambda \cdot 0). \end{aligned} \tag{4}$$

By the zero fixed cost condition,

$$c(x) = \lambda c(x) + (1-\lambda)c(0) + (1-\lambda)c(x) + \lambda c(0).$$

Then, (4) means that

$$\lambda c(x) + (1-\lambda)c(0) \leq c(\lambda x + (1-\lambda)\cdot 0). \tag{5}$$

or

$$(1-\lambda)c(x) + \lambda c(0) \leq c((1-\lambda)x + \lambda \cdot 0). \tag{6}$$

holds. By the assumption that $c(x)$ is convex throughout the domain or concave throughout the domain, if either of (5) and (6) holds, $c(x)$ is concave.

Since λ is arbitrary, this result implies that sub-additivity of $c(x)$ means its concavity in an interval including the origin.

□

3 Concluding Remark

We usually assume that a cost function is convex throughout the domain or concave throughout the domain. Therefore, if there is no fixed cost, we can use interchangeably convexity and super-additivity of a cost function, and use interchangeably concavity and sub-additivity of a cost function.

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