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Equivalence of the HEX game theorem and the Arrow impossibility theorem

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Abstract

Gale [D. Gale, The game of HEX and the Brouwer fixed-point theorem, American Mathematical Monthly 86 (1979) 818–827] has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this paper we will show that under some assumptions about marking rules of HEX games, the HEX game theorem is equivalent to the Arrow impossibility theorem of social choice theory that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator. We assume that individual preferences over alternatives are strong (or linear) orders, that is, the individuals are not indifferent about any pair of alternatives.

Keywords: The HEX game theorem; The Arrow impossibility theorem

1. Introduction

Gale [3] has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this paper we will show that under some assumptions about marking rules of HEX games, the HEX game theorem is equivalent to the Arrow impossibility theorem of social choice theory [1] that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator. We assume that individual preferences over alternatives are strong (or linear) orders, that is, the individuals are not indifferent about any pair of alternatives.

In the next section according to Gale [3] we present an outline of the HEX game. In Section 3, we will show that the HEX game theorem implies the Arrow impossibility theorem. And in Section 4, we will show that the Arrow impossibility theorem implies the HEX game theorem.

2. The HEX game

According to Gale [3] we present an outline of the HEX game. Fig. 1(a) represents a 6×6 HEX board. Generally a HEX game is represented by an $n \times n$ HEX board where n is a finite positive integer. The rules

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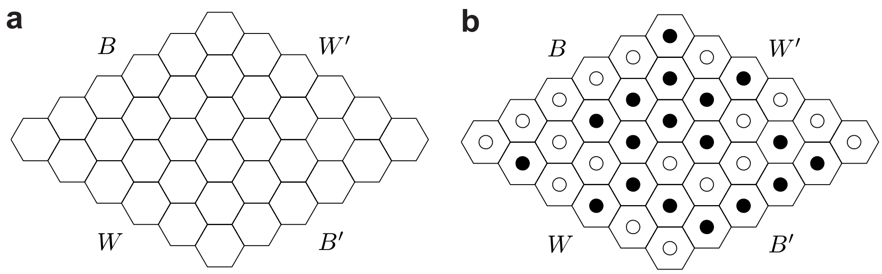

Fig. 1. HEX game.

of the game are as follows. Two players (called Mr. W and Mr. B) move alternately, marking any previously unmarked hexagon or tile with a white (by Mr. W) or a black (by Mr. B) circle respectively. The game has been won by Mr. W (or Mr. B) if he has succeeded in marking a connected set of tiles which meets the boundary regions W and W' (or, B and B'). A set S of tiles is connected if any two members of the set h and h' can be joined by a path $P = (h = h^1, h^2, \ldots, h^m = h')$ where h^i and h^{i+1} are adjacent. Fig. 1(b) represents a HEX game which has been won by Mr. B.

About the HEX game Gale [3] has shown the following theorem.

Theorem 1 (The HEX game theorem). *If every tile of the HEX board is marked by either a white or a black circle, then there is a path connecting regions W and W', or a path connecting regions B and B'.*

Actually he has shown the theorem that any hex game can never end in a draw, and there always exists at least one winner. But, from his intuitive explanation using the following example of river and dam, it is clear that there exists only one winner of any hex game.

> Imagine that B and B' regions are portions of opposite banks of the river which flow from W region to W' region, and that Mr. B is trying to build a dam by putting down stones. He will have succeeded in damming the river if and only if he has placed his stones in a way which enables him to walk on them from one bank (B region) to the other (B' region).

The proof of Theorem 1 and also the above intuitive argument do not depend on the rule "two players move *alternately*". Therefore, this theorem is valid for any marking rule.

Fig. 2(a) is obtained by plotting the center of each hexagon, and connecting these centers by lines. Rotating this graph 45° in anticlockwise direction, we obtain Fig. 2(b). It is an equivalent representation of the HEX board depicted in Fig. 1(a). W and W' represent the regions of Mr. W, and B and B' represent the regions of Mr. B. We call it a square HEX board, and call a game represented by a square HEX board a *square HEX*

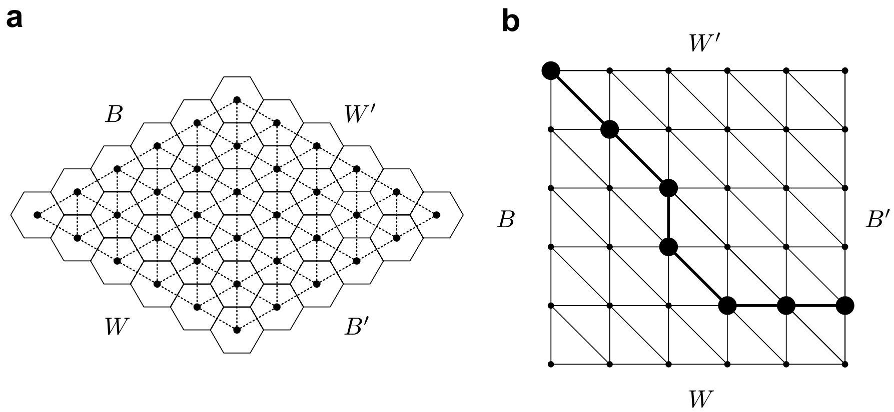

Fig. 2. Square HEX game and winning path.

game. In Fig. 2(b) we depict an example of winning marking by Mr. B. It corresponds to the marking pattern in Fig. 1(b). A set of marked vertices which represents one player's victory is called a *winning path*.

3. The HEX game theorem implies the Arrow impossibility theorem

There are m ($\geqslant 3$) alternatives and n ($\geqslant 2$) individuals. m and n are finite positive integers. The set of individuals is denoted by N. Denote individual i's preference by p_i. A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}$ $(=(p_1, p_2, \ldots, p_n))$. The set of profiles is denoted by $\mathscr{P}^n$. The alternatives are represented by x_i, $i = 1, 2, \ldots, m$. Individual preferences over the alternatives are strong (or linear) orders, that is, individuals strictly prefer one alternative to another, and are not indifferent about any pair of alternatives. We assume the free triple property, that is, for each set of three alternatives individual preferences are never restricted.

We consider a binary social choice rule which determines a social preference corresponding to each profile. Binary social choice rules must satisfy the conditions of *transitivity*, *Pareto principle* and *independence of irrelevant alternatives* (*IIA*). Such binary social choice rules are called *social welfare functions*. The meanings of these conditions are as follows.

Transitivity. If, according to a social welfare function, the society prefers an alternative x_i to another alternative x_j, and prefers x_j to another alternative x_k, then the society must prefer x_i to x_k.
Pareto principle. When all individuals prefer x_i to x_j, the society must prefer x_i to x_j.
Independence of irrelevant alternatives (*IIA*). The social preference about every pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

From Lemma 1 of [2] we know that if individual preferences are linear orders, then the social preference is also a linear order under the conditions of transitivity, Pareto principle and IIA.

The Arrow impossibility theorem states that there exists no social welfare function which has no dictator, or in other words there exists a dictator for any social welfare function. A dictator is an individual whose strict preference always coincides with the social preference.

According to Sen [4] we define the following terms:

Almost decisiveness. If, when all individuals in a group G prefer an alternative x_i to another alternative x_j, and the other individuals (individuals in $N\backslash G$) prefer x_j to x_i, the society prefers x_i to x_j, then G is *almost decisive* for x_i against x_j.
Decisiveness. If, when all individuals in a group G prefer an alternative x_i to another alternative x_j, the society prefers x_i to x_j regardless of the preferences of the other individuals, then G is *decisive* for x_i against x_j.

G may consists of one individual. By Pareto principle N is almost decisive and decisive about every pair of alternatives. If for a social welfare function an individual is decisive about every pair of alternatives, then he is the *dictator* of the social welfare function.

Sen [4] and Suzumura [5] have shown the following result.

Lemma 1 (Lemma 3*a in [4] and Dictator Lemma in [5]). *If one individual is almost decisive for one alternative against another alternative, then he is the dictator of the social welfare function.*

This lemma holds under the conditions of transitivity, Pareto principle and IIA. The conclusion of this lemma is also valid in the case where not an individual but a group of individuals is almost decisive for one alternative against another alternative. Thus, the following lemma is derived.

Lemma 2. *If a group of individuals G is almost decisive for one alternative against another alternative, then this group is decisive about every pair of alternatives.*

Now we confine us to a subset of profiles $\overline{\mathscr{P}}^n$ such that all individuals prefer three alternatives x_1, x_2 and x_3 to all other alternatives. Pareto principle implies that at all such profiles the society also prefers x_1, x_2 and x_3

to all other alternatives. We denote individual preferences about x_1, x_2 and x_3 in this subset of profiles as follows:

$$p^1 = (123), \quad p^2 = (132), \quad p^3 = (312), \quad p^4 = (321), \quad p^5 = (231), \quad p^6 = (213),$$

$p^1 = (123)$ represents all preferences such that an individual prefers x_1 to x_2 to x_3 to all other alternatives, and so on. Although we confine our arguments to such a subset of profiles, Lemma 1 with IIA ensures that an individual who is almost decisive about a pair of alternatives for this subset of profiles is the dictator for all profiles.

From Lemma 2 for the profiles in $\overline{\mathscr{P}}^n$ we obtain the following result.

Lemma 3. *If two groups G and G', which are not disjoint, are almost decisive about a pair of alternatives, then their intersection $G \cap G'$ is decisive about every pair of alternatives.*

Proof. By Lemma 2 G and G' are decisive about every pair of alternatives. For three alternatives x_1, x_2 and x_3 we consider the following profile in $\overline{\mathscr{P}}^n$:

(1) Individuals in $G \backslash (G \cap G')$ prefer x_3 to x_1 to x_2.
(2) Individuals in $G' \backslash (G \cap G')$ prefer x_2 to x_3 to x_1.
(3) Individuals in $G \cap G'$ prefer x_1 to x_2 to x_3.
(4) Individuals in $N \backslash (G \cup G')$ prefer x_3 to x_2 to x_1.

By the decisiveness of G and G' and transitivity the society must prefer x_1 to x_2 to x_3. Since only individuals in $G \cap G'$ prefer x_1 to x_3 and all other individuals prefer x_3 to x_1, $G \cap G'$ is almost decisive for x_1 against x_3 under IIA. From Lemma 2 it is decisive about every pair of alternatives. □

Further we confine us to a subset of $\overline{\mathscr{P}}^n$ such that all but one individual have the same preferences, and consider a HEX game between one individual (denoted by individual k) and the set of individuals other than k. Representative profiles are denoted by (p^i_k, p^j_{-k}), $i = 1, \ldots, 6$, $j = 1, \ldots, 6$, where p^i_k is individual k's preference and p^j_{-k} denotes the common preference of the individuals other than k. We relate these profiles to the vertices of a 6×6 square HEX board as depicted in Fig. 3. There are 36 vertices in this HEX board. It represents a square HEX game. k and k' represent individual k's regions, and $-k$ and $-k'$ represent the regions of the set of individuals other than k.

We consider the following marking and winning rules of the square HEX game:

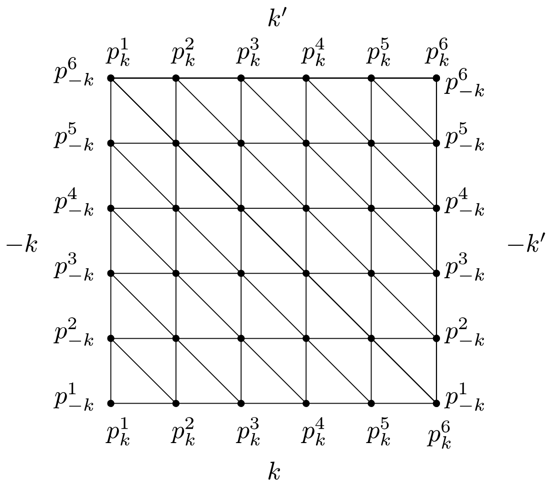

Fig. 3. HEX game representing profiles.

(1) At a profile represented by a vertex of a square HEX board, if the society's most preferred alternative is the same as that of individual k and different from that of the individuals other than k, then this vertex is marked by a white circle; conversely if the society's most preferred alternative is the same as that of the individuals other then k and different from that of individual k, then this vertex is marked by a black circle. Hereafter we abbreviate the most preferred alternative by MPA.
(2) At a profile, if the society's MPA is different from the MPA of individual k and the MPA of the individuals other than k, or the MPAs of all individuals are the same, then the vertex which corresponds to this profile is randomly marked by a white or a black circle.
(3) The game has been won by individual k (or the set of individuals other than k) if he has (or they have) succeeded in marking a connected set of vertices which meets the boundary regions k and k' (or $-k$ and $-k'$).

A square HEX game is equivalent to the original HEX game. Therefore, there exists one winner for any marking rule. Now we show the following result.

Theorem 2. *The HEX game theorem implies the existence of a dictator for any social welfare function.*

Proof. Since diagonal vertices are not connected, the diagonal path

$$((p_k^1, p_{-k}^1), (p_k^2, p_{-k}^2), \ldots, (p_k^6, p_{-k}^6))$$

cannot be a winning path. Consider a profile $(p_k^1, p_{-k}^6) = ((123), (213))$. By Pareto principle x_3 is not the society's MPA. Suppose that at this profile the society's MPA is x_2 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_2:

$$(p_k^1, p_{-k}^5), (p_k^2, p_{-k}^6).$$

The fact that the society's MPA at a profile (p_k^2, p_{-k}^6) is x_2, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_2:

$$(p_k^3, p_{-k}^5), (p_k^4, p_{-k}^5), (p_k^4, p_{-k}^6).$$

Similarly consider a profile $(p_k^3, p_{-k}^2) = ((312), (132))$. By Pareto principle x_2 is not the society's MPA. Suppose that at this profile the society's MPA is x_1 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_1:

$$(p_k^3, p_{-k}^1), (p_k^4, p_{-k}^2).$$

The fact that the society's MPA at a profile (p_k^4, p_{-k}^2) is x_1, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_1:

$$(p_k^5, p_{-k}^1), (p_k^6, p_{-k}^1), (p_k^6, p_{-k}^2).$$

Similarly consider a profile $(p_k^5, p_{-k}^4) = ((231), (321))$. By Pareto principle x_1 is not the society's MPA. Suppose that at this profile the society's MPA is x_3 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_3:

$$(p_k^5, p_{-k}^3), (p_k^6, p_{-k}^4).$$

The fact that the society's MPA at a profile (p_k^6, p_{-k}^4) is x_3, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_3:

$$(p_k^1, p_{-k}^3), (p_k^2, p_{-k}^3), (p_k^2, p_{-k}^4).$$

The vertices which correspond to all of these profiles are marked by black circles. Then, even when all other vertices are marked by white circles, we obtain a marking pattern of a square HEX board as depicted in Fig. 4. The set of individuals other than k is the winner of this HEX game. Therefore, for individual k to be the winner of a square HEX game, the society's MPA must coincide with that of individual k at least at one of three profiles (p_k^1, p_{-k}^6), (p_k^3, p_{-k}^2) and (p_k^5, p_{-k}^4). It means that individual k must be almost decisive about at least one pair of alternatives, and then by Lemma 1 he is the dictator.

If for all k, ($k = 1, 2, \ldots, n$), individual k is not the winner of any square HEX game between individual k and the set of individuals other than k, then each set of individuals excluding one individual is the winner of each square HEX game. By Lemma 3 every nonempty intersection of the sets of individuals excluding one individual is decisive. Then, the intersection of $N\backslash\{1\}, N\backslash\{2\}, \ldots, N\backslash\{n-1\}$ is decisive. But $(N\backslash\{1\}) \cap (N\backslash\{2\}) \cap \cdots (N\backslash\{n-1\}) = \{n\}$. Thus, individual n is the dictator. Therefore, the HEX game theorem implies the existence of a dictator for any social welfare function. □

By this theorem the HEX game theorem implies the Arrow impossibility theorem.

4. The Arrow impossibility theorem implies the HEX game theorem

Next we will show that the Arrow impossibility theorem implies the HEX game theorem under an interpretation of dictator. Similarly to the previous section, we confine us to a subset of profiles such that all individuals prefer three alternatives x_1, x_2 and x_3 to all other alternatives, and the preferences of individuals other than one individual (denoted by k) are the same. And we consider a square HEX game between individual k and the set of individuals other than k. The dictator of a social welfare function is interpreted as an individ-

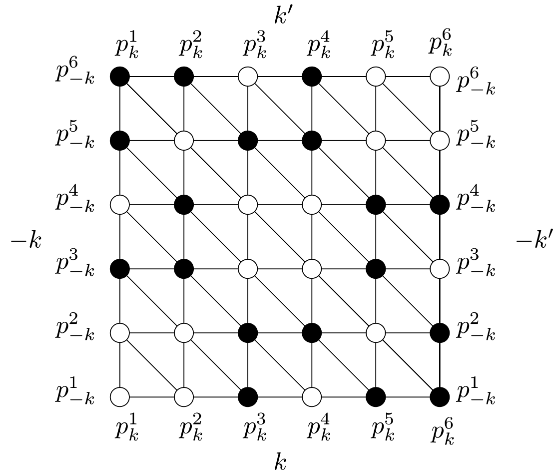

Fig. 4. Winning path of a square HEX game.

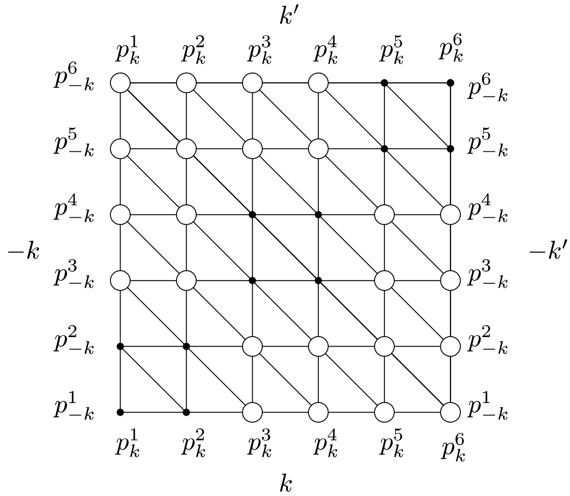

Fig. 5. HEX game won by individual k.

ual who can determine the MPA of the society when his MPA and that of the other individuals are different, and in a HEX game he can mark tiles with his color in such cases. We denote a vertex of a square HEX board which corresponds to a profile (p_k^i, p_{-k}^j) simply by (p_k^i, p_{-k}^j).

First, consider the case where individual k is the dictator of a social welfare function. Then, the following vertices are marked by white circles:

$$(p_k^1, p_{-k}^3), (p_k^1, p_{-k}^4), (p_k^1, p_{-k}^5), (p_k^1, p_{-k}^6), (p_k^2, p_{-k}^3), (p_k^2, p_{-k}^4), (p_k^2, p_{-k}^5), (p_k^2, p_{-k}^6),$$
$$(p_k^3, p_{-k}^1), (p_k^3, p_{-k}^2), (p_k^3, p_{-k}^5), (p_k^3, p_{-k}^6), (p_k^4, p_{-k}^1), (p_k^4, p_{-k}^2), (p_k^4, p_{-k}^5), (p_k^4, p_{-k}^6),$$
$$(p_k^5, p_{-k}^1), (p_k^5, p_{-k}^2), (p_k^5, p_{-k}^3), (p_k^5, p_{-k}^4), (p_k^6, p_{-k}^1), (p_k^6, p_{-k}^2), (p_k^6, p_{-k}^3), (p_k^6, p_{-k}^4).$$

We obtain Fig. 5. Unmarked vertices, where the MPAs of all individuals are the same, should be randomly marked. Clearly individual k is the winner of this HEX game.

Next, consider the case where the dictator of a social welfare function is included in the set of individuals other than k. Then, all of the above vertices are marked by black circles, and the set of individuals other than k is the winner of the HEX game.

Therefore, the existence of a dictator for a social welfare function implies the existence of a winner for a HEX game, and we obtain

Theorem 3. *The Arrow impossibility theorem and the HEX game theorem are equivalent.*

5. Concluding remarks

We have considered the relationship between the HEX game theorem and the Arrow impossibility theorem, and have shown their equivalence. In this paper we have assumed that individual preferences over alternatives are strong (or linear) orders. We are now proceeding research on extension of the result of this paper to the case where individual preferences over alternatives are weak orders, that is, they may be indifferent about any pair of two alternatives.

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