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On the topological equivalence of the Arrow impossibility theorem and Amartya Sen's liberal paradox

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Abstract

We will show that the Arrow impossibility theorem for binary social choice rules that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA), and has no dictator, and Amartya Sen's liberal paradox for binary social choice rules that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism are topologically equivalent using elementary tools of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by [Y. Baryshnikov, Unifying impossibility theorems: a topological approach, Advances in Applied Mathematics 14 (1993) 404–415]. Also we will show that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the *non-surjectivity of individual inclusion mappings*.

Keywords: Simplicial complexes; Simplicial mappings; Homology groups; Homomorphisms; Arrow impossibility theorem; Amartya Sen's liberal paradox

1. Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky [6]. Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky [5,7], Candeal and Indurain [4], Koshevoy [9], Lauwers [11], Weinberger [15], and so on. On the other hand, Baryshnikov [2,3] have presented a topological approach to the Arrow impossibility theorem (or general possibility theorem) in a discrete framework of social choice.[1]

We will show that the Arrow impossibility theorem for binary social choice rules that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA), and has no dictator, and Amartya Sen's liberal paradox for binary social choice rules that there exists no

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[1] About surveys and basic results of topological social choice theories, see Mehta [12] and Lauwers [10].

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binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism are topologically equivalent using elementary tools of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings. Also we will show that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the *non-surjectivity of individual inclusion mappings*. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov [2].

In the next section we present expressions of binary social choice rules by simplicial complexes and simplicial mappings. In Section 3, we will prove the main results of this paper.

2. The expressions of social choice problems by simplicial complexes and simplicial mappings

There are m alternatives of a social problem, $x_1, x_2, \ldots, x_m$ ($m \geqq 3$), and n individuals ($n \geqq 2$). m and n are finite integers. Individual preferences over these alternatives are complete, transitive and asymmetric.

A social choice rule which we will consider is a rule that determines a social preference about each pair of alternatives corresponding to a combination of individual preferences. We call such a social choice rule a *binary social choice rule*. The social preference should be complete, but may be or may not be transitive. As usual we assume the universal domain condition for social choice rules. We call a combination of individual preferences a *profile*. The profiles are denoted by $\mathbf{p}$, $\mathbf{p}'$ and so on.

We will consider two social choice problems about binary social choice rules.

(1) **(Amartya Sen's liberal paradox):** The liberal paradox by Amartya Sen [13] states that there exists no binary social choice rule which satisfies *acyclicity*, *Pareto principle* and the *minimal liberalism*. The means of these conditions are as follows.
Acyclicity: If the society (strictly) prefers x_i to x_j, and (strictly) prefers x_j to x_k, then it should prefer x_i to x_k or be indifferent between them. It is weaker than transitivity which requires that the society (strictly) prefers x_i to x_k.
Pareto principle: If all individuals prefer an alternative x_i to another alternative x_j, then the society prefers x_i to x_j.
Minimal liberalism: At least two individuals, denoted by A and B, are decisive for some pairs of alternatives in both directions in the sense described in the following Assumption 1.

In what follows as the condition of the minimal liberalism we assume

Assumption 1. If individual A prefers x_1 to x_3 (or prefers x_3 to x_1), then the society prefers x_1 to x_3 (or prefers x_3 to x_1). And if individual B prefers x_2 to x_4 (or prefers x_4 to x_2), then the society prefers x_2 to x_4 (or prefers x_4 to x_2).

Other individuals are not necessarily decisive. We can proceed the arguments in a similar manner based on other assumptions about the minimal liberalism by permuting or renaming alternatives. We abbreviate the problem of the liberal paradox as LP.

(2) **(The Arrow impossibility theorem):** The Arrow impossibility theorem [1] states that there exists no binary social choice rule which satisfies *transitivity*, *Pareto principle* and *independence of irrelevant alternatives (IIA)*, and *has no dictator*, or in other words there exists the dictator for any binary social choice rule which satisfies transitivity, Pareto principle and IIA. The dictator for a binary social choice rule is an individual such that whenever he (strictly) prefers one alternative (denoted by x) to another alternative (denoted by y), the society also (strictly) prefers x to y. The meanings of two conditions, transitivity and IIA, are as follows.
Transitivity: If the society (strictly) prefers x_i to x_j, and (strictly) prefers x_j to x_k, then the society should (strictly) prefer x_i to x_k.
Independence of irrelevant alternatives (IIA): The society's preference about any pair of two alternatives depends only on individual preferences about these alternatives.

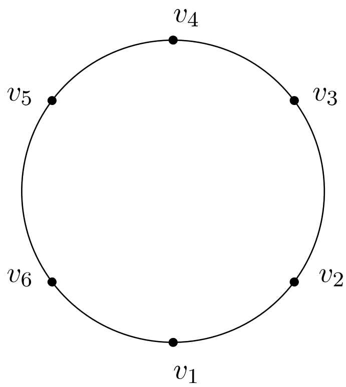

Fig. 1. S_i^1.

We abbreviate the problem of the Arrow impossibility theorem as AR. Pareto principle for AR is the same as that for LP.

We draw a circumference which represents the set of individual preferences by connecting $m!$ vertices $v_1, v_2, \ldots, v_{m!}$ by arcs.[2] For example, in the case of four alternatives, these vertices mean the following preferences:

$$\begin{array}{llllll} v_1 : (1234), & v_2 : (1243), & v_3 : (1423), & v_4 : (1432), & v_5 : (1342), & v_6 : (1324), \\ v_7 : (2134), & v_8 : (2143), & v_9 : (2413), & v_{10} : (2431), & v_{11} : (2341), & v_{12} : (2314), \\ v_{13} : (3124), & v_{14} : (3142), & v_{15} : (3412), & v_{16} : (3421), & v_{17} : (3241), & v_{18} : (3214), \\ v_{19} : (4123), & v_{20} : (4132), & v_{21} : (4312), & v_{22} : (4321), & v_{23} : (4231), & v_{24} : (4213). \end{array}$$

We denote a preference such that an individual prefers x_1 to x_2 to x_3 to x_4 by (1234), and so on. Notations for the cases with different number of alternatives are similar. Generally $v_1 \sim v_{(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_1, $v_{(m-1)!+1} \sim v_{2(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_2, and so on. And v_1 is a preference such that an individual prefers x_1 to x_2 to x_3 to $\cdots$ to x_m. It is denoted by $(123 \cdots m)$. $v_{(m-1)!+1}$ is a preference such that an individual prefers x_2 to x_1 to x_3 to x_4 to $\cdots$ to x_m, which is denoted by $(2134 \cdots m)$.

Denote this circumference by S_i^1. S_i^1 in the case of three alternatives is depicted in Fig. 1. The set of profiles of the preferences of n individuals is represented by the product space $S_i^1 \times \cdots \times S_i^1$ (n times). It is denoted by $(S_i^1)^n$. The one-dimensional homology group of S_i^1 is isomorphic to the group of integers $\mathbb{Z}$, that is, $H_1(S_i^1) \cong \mathbb{Z}$. And the one-dimensional homology group of $(S_i^1)^n$ is isomorphic to the direct product of n groups of integers $\mathbb{Z}^n$, that is, we have $H_1((S_i^1)^n) \cong \mathbb{Z}^n$. It is proved, for example, using the Mayer–Vietoris exact sequences.[3]

The social preference is also represented by a circumference depicted in Fig. 2. This circumference is drawn by connecting three vertices, w_1, w_2 and w_3 by arcs. For LP these vertices mean the following social preferences:

(1) w_1: social preferences such that the society prefers x_4 to all other alternatives,
(2) w_3: social preferences such that the society prefers x_3 to all other alternatives,
(3) w_2: all other social preferences.

[2] $m!$ denotes the factorial of m.

$$m! = \prod_{j=1}^{m} j = m(m-1)(m-2) \times \cdots \times 2 \times 1.$$

[3] About homology groups and the Mayer–Vietoris exact sequences we referred to Tamura [14] and Komiya [8].

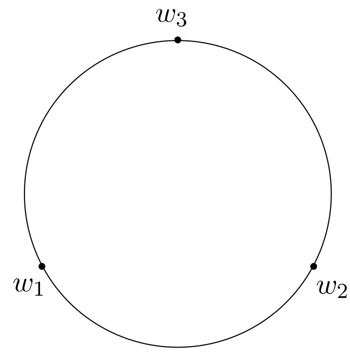

Fig. 2. S^1.

Similarly for AR these vertices mean the following social preferences:

(1) w_1: social preferences such that the society prefers x_4 to all other alternatives,
(2) w_3: social preferences such that the society prefers x_3 to all other alternatives,
(3) w_2: all other social preferences.

That is, the vertices w_1 and w_3 denote the same social preferences for LP and AR, and the set of social preferences expressed by w_2 for AR is the proper subset of the set of social preferences expressed by w_2 for LP because the social preference are required to satisfy transitivity in AR, but in LP we require only acyclicity.

We call this circumference S^1. The one-dimensional homology group of S^1 is also isomorphic to $\mathbb{Z}$, that is, $H_1(S^1) \cong \mathbb{Z}$.

Binary social choice rules are simplicial mappings. Binary social choice rules in AR and LP are denoted by $f : (S_i^1)^n \to S^1$. Two adjacent vertices of S_i^1 span a one-dimensional simplex. And any pair of two vertices of S^1 spans a one-dimensional simplex. Thus, f is a simplicial mapping, and we can define the homomorphism of homology groups induced by f.

We define an inclusion mapping from S_i^1 to $(S_i^1)^n$ by $\Delta : S_i^1 \to (S_i^1)^n$ under the assumption that all individuals have the same preferences, and define an inclusion mapping when the profile of preferences of individuals other than one individual (denoted by i) is fixed at some profile by $i_i : S_i^1 \to (S_i^1)^n$. The homomorphisms of homology groups induced by these inclusion mappings are as follows:

$$\Delta_* : \mathbb{Z} \to \mathbb{Z}^n : h \to (h, h, \ldots, h), \quad h \in \mathbb{Z},$$
$$i_{i*} : \mathbb{Z} \to \mathbb{Z}^n : h \to (0, \ldots, 0, h, 0, \ldots, 0), \ h \in \mathbb{Z} \text{ (only the } i\text{th component is } h).$$

From these definitions we obtain the following relation about Δ_* and i_{i*} at any profile:

$$\Delta_* = \sum_{i=1}^{n} i_{i*}. \tag{1}$$

Let the homomorphism of homology groups induced by f be $f_* : (\mathbb{Z})^n \to \mathbb{Z}$.

Binary social choice rules for different profiles are homotopic. f for a fixed profile of the preferences of individuals other than i (denoted by $f|_{\mathbf{p}_{-i}}$) and f for another fixed profile of the preferences of individuals other than i (denoted by $f|_{\mathbf{p}'_{-i}}$) are homotopic. Thus, the homomorphisms of homology groups induced by them are isomorphic. Denote two profiles of individuals other than i by $\mathbf{p}_{-i}$ and $\mathbf{p}'_{-i}$. Then, the homotopy between $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ is $f_t = \frac{tf|_{\mathbf{p}_{-i}} + (1-t)f|_{\mathbf{p}'_{-i}}}{|tf|_{\mathbf{p}_{-i}} + (1-t)f|_{\mathbf{p}'_{-i}}|}$ $(0 \leqq t \leqq 1)$. It is well defined since $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ are not antipodal.

The composite function of i_i and f is denoted by $f \circ i_i : S_i^1 \to S^1$, and its induced homomorphism of homology groups satisfies $(f \circ i_i)_* = f_* \circ i_{i*}$, for all i. The composite function of Δ and f is denoted by $f \circ \Delta : S_i^1 \to S^1$, and its induced homomorphism of homology groups satisfies $(f \circ \Delta)_* = f_* \circ \Delta_*$. From (1) we obtain

$$(f \circ \Delta)_* = \sum_{i=1}^{n} (f \circ i_i)_*. \tag{2}$$

3. The main results

For binary social choice rules in AR we define the following concept:

Weak monotonicity. For two alternatives x_i and x_j, suppose that at profile p the society prefers x_i to x_j. And suppose that individuals, who prefer x_i to x_j at p, prefer x_i to x_j at another profile p'. Then, the society prefers x_i to x_j at p'.

We show the following result.

Lemma 1. *Any binary social choice rule in AR which satisfies transitivity, Pareto principle and IIA satisfies the weak monotonicity.*

Proof. We use notations in the definition of the weak monotonicity. Let x_k be an arbitrary alternative other than x_i and x_j.

Suppose that individuals, who prefer x_i to x_j at p, prefer x_i to x_j to x_k at another profile p'', and individuals, who prefer x_j to x_i at p, prefer x_j to x_k to x_i at p''.

And suppose that individuals, who prefer x_i to x_j at p, prefer x_i to x_k to x_j at another profile p^*, and individuals, who prefer x_j to x_i at p, prefer x_k to x_i and prefer x_k to x_j at p^* (their preferences about x_i and x_j are not specified).

By transitivity, Pareto principle and IIA the society prefers x_i to x_j to x_k at p''. Again by transitivity, Pareto principle and IIA (about x_i and x_k) the society prefers x_i to x_k to x_j at p^*. Then, IIA implies that the society prefers x_i to x_j so long as individuals, who prefer x_i to x_j at p, prefer x_i to x_j at an arbitrary profile p'. □

Next we show the following lemma which will be used below:

Lemma 2. *Suppose that a binary social choice rule satisfies transitivity, Pareto principle, IIA, and has no dictator. If the preference of one individual (denoted by i) is $v_{(m-1)!+1}$ and the preferences of all other individuals are v_1, then the most preferred alternative for the society is x_1.*

Proof. Note that $v_{(m-1)!+1}$ represents a preference $(2134 \cdots m)$, and v_1 represents a preference $(123 \cdots m)$. By Pareto principle the society prefers x_1 and x_2 to all other alternatives. It may prefer x_1 to x_2, or x_2 to x_1.[4] But we can show that if the society prefers x_2 to x_1, individual i is the dictator. Assume that the society prefers x_2 to x_1 to all other alternatives. By the weak monotonicity the society prefers x_2 to x_1 so long as individual i prefers x_2 to x_1. Then, we say that individual i is *decisive* for x_2 against x_1. Let x_j and x_k ($x_k \neq x_j$) be alternatives other than x_1 and x_2, and consider the following profile:

(1) Individual i prefers x_k to x_2 to x_1 to x_j.
(2) Other individuals prefer x_1 to x_j to x_k to x_2.

By the weak monotonicity (or IIA) the society should prefer x_2 to x_1. And by Pareto principle the society should prefer x_1 to x_j, and prefer x_k to x_2. Then, transitivity implies that the society prefers x_k to x_j. The weak

[4] From Lemma 1 of Baryshnikov [2] we know that if individual preferences are strict orders, then the social preference is also a strict order under transitivity, Pareto principle and IIA.

monotonicity implies that the society prefers x_k to x_j so long as individual i prefers x_k to x_j, and individual i is decisive for x_k against x_j. Note that x_j and x_k are arbitrary. Next consider the following profile:

(1) Individual i prefers x_2 to x_k to x_j.
(2) Other individuals prefer x_j to x_2 to x_k.

By the weak monotonicity the society should prefer x_k to x_j. And by Pareto principle the society should prefer x_2 to x_k. Then, transitivity implies that the society prefers x_2 to x_j. The weak monotonicity implies that the society prefers x_2 to x_j so long as individual i prefers x_2 to x_j, and individual i is decisive for x_2 against x_j. Consider the following profile:

(1) Individual i prefers x_k to x_j to x_2.
(2) Other individuals prefer x_j to x_2 to x_k.

By the weak monotonicity the society should prefer x_k to x_j. And by Pareto principle the society should prefer x_j to x_2. Then, transitivity implies that the society prefers x_k to x_2. The weak monotonicity implies that the society prefers x_k to x_2 so long as individual i prefers x_k to x_2, and individual i is decisive for x_k against x_2. By similar procedures we can show that individual i is decisive for x_1 against x_j, and is decisive for x_k against x_1. Finally consider the following profile:

(1) Individual i prefers x_1 to x_k to x_2.
(2) Other individuals prefer x_2 to x_1 to x_k.

By the weak monotonicity the society should prefer x_k to x_2. And by Pareto principle the society should prefer x_1 to x_k. Then, transitivity implies that the society prefers x_1 to x_2. The weak monotonicity implies that the society prefers x_1 to x_2 so long as individual i prefers x_1 to x_2, and individual i is decisive for x_1 against x_2. Therefore, individual i is the dictator, and we must assume that the society prefers x_1 to all other alternatives when the preference of individual i is $v_{(m-1)!+1}$ and the preferences of individuals other than i are v_1. □

In both AR and LP cases, by Pareto principle we obtain the correspondences from the vertices of S_i^1 to the vertices of S^1 by $f \circ \Delta$ as follows:

$$v_1 \sim v_{2(m-1)!} \longrightarrow w_2, \quad v_{2(m-1)!+1} \sim v_{3(m-1)!} \longrightarrow w_3, \quad v_{3(m-1)!+1} \sim v_{4(m-1)!} \longrightarrow w_1.$$

All other vertices correspond to w_2. Sets of one-dimensional simplices included in S_i^1 which are one-dimensional cycles are only the following z and its counterpart $-z$:

$$z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m!-1}, v_{m!} \rangle + \langle v_{m!}, v_1 \rangle.$$

Since S_i^1 does not have a two-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ \Delta)_*$ to the following z':

$$z' = \langle w_2, w_3 \rangle + \langle w_3, w_1 \rangle + \langle w_1, w_2 \rangle.$$

This is a cycle of S^1. Therefore, we have

$$(f \circ \Delta)_* \neq 0. \tag{3}$$

Now we show the following lemma.

Lemma 3

(1) *If a binary social choice rule satisfies acyclicity, Pareto principle and the minimal liberalism described in Assumption 1, then we obtain*

$$(f \circ i_i)_* = 0 \quad \textit{for all } i. \tag{4}$$

(2) *If a binary social choice rule satisfies transitivity, Pareto principle and IIA, then we obtain (4).*

Proof

(1) First we show $(f \circ i_i)_* = 0$ for individual A and B. Consider the case of individual B. From Assumption 1 and Pareto principle, the correspondences from the preference of individual B to the social preference when the preference of every other individual (including individual A) is fixed at v_1 are obtained as follows:

$$v_1 \sim v_{(m-1)!} \to w_2, \quad v_{(m-1)!+1} \sim v_{m!} \to w_1 \text{ or } w_2.$$

In this case x_3 cannot be the most preferred alternative for the society.
Sets of one-dimensional simplices included in S^1_i for individual B (denoted by S^1_B) which are one-dimensional cycles are only the following z and its counterpart $-z$:

$$z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m!-1}, v_{m!} \rangle + \langle v_{m!}, v_1 \rangle.$$

Since S^1_B does not have a two-dimensional simplex, z is a representative element of homology classes of S^1_B. z is transferred by $(f \circ i_B)_*$, which is $(f \circ i_i)_*$ for individual B, to the following z' in S^1:

$$z' = \langle w_2, w_1 \rangle + \cdots + \langle w_1, w_2 \rangle = 0 \quad \text{or} \quad z' = \langle w_2, w_2 \rangle = 0.$$

This is not a cycle. Therefore, we get $(f \circ i_B)_* = 0$. Similarly we can show $(f \circ i_A)_* = 0$.[5] Next we show $(f \circ i_i)_* = 0$ for any individual (denoted by i) other than A and B. From Assumption 1 and Pareto principle, the correspondences from the preference of individual i to the social preference when the preference of every other individual (including individual A and B) is fixed at v_1 are obtained as follows:

$$v_1 \sim v_{m!} \to w_2.$$

Because x_3 and x_4 cannot be the most preferred alternative for the society. Then, we obtain $(f \circ i_i)_* = 0$ for all i other than A and B.

(2) By Pareto principle when the preference of every individual other than i is fixed at v_1, the correspondences from the preference of individual i to the social preference from v_1 to $v_{(m-1)!}$ are as follows:

$$v_1 \sim v_{(m-1)!} \to w_2.$$

From Lemma 2 the correspondence from $v_{(m-1)!+1}$ to the social preference is as follow:

$$v_{(m-1)!+1} \to w_2.$$

Consider another profile at which the preference of individual i changes to $(234 \cdots m1)$. By Pareto principle and the weak monotonicity (about x_1 and x_2) the society prefers x_1 to all other alternatives. Further the weak monotonicity implies that the society prefers x_1 to all other alternatives so long as the most preferred alternative for all individuals other than i is x_1 regardless of the preference of individual i. Thus, we obtain the following correspondences:

$$v_{(m-1)!+2} \sim v_{m!} \to w_2.$$

Sets of one-dimensional simplices included in S^1_i which are one-dimensional cycles are only the following z and its counterpart $-z$:

$$z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m!-1}, v_{m!} \rangle + \langle v_{m!}, v_1 \rangle.$$

Since S^1_i does not have a two-dimensional simplex, z is a representative element of homology classes of S^1_i. z is transferred by $(f \circ i_i)_*$ to the following z':

$$z' = \langle w_2, w_2 \rangle = 0.$$

Therefore, we have $(f \circ i_i)_* = 0$ for all i. □

The conclusion of this lemma contradicts (2) and (3) for both LP and AR. Therefore, we have shown the following theorem.

[5] $(f \circ i_A)_*$ is $(f \circ i_i)_*$ for individual A. In this case x_4 cannot be the most preferred alternative for the society.

Theorem 1

(1) *There exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism.*
(2) *There exists no binary social choice rule which satisfies transitivity, Pareto principle and IIA, and has no dictator.*

We call the property expressed in (4) the *non-surjectivity of individual inclusion mappings*. Then, the above two theorems are special cases of the following theorem:

Theorem 2. *There exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.*

From (3) Pareto principle implies the *surjectivity of the diagonal mapping* $(f \circ \Delta)_* \neq 0$, for binary social choice rules. Thus, this theorem is rewritten as follows:

> There exists no binary social choice rule which satisfies the *surjectivity of the diagonal mapping* and the *non-surjectivity of individual inclusion mappings.*

4. Concluding remarks

We have shown the topological equivalence of the Arrow impossibility theorem that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives, and has no dictator, and Amartya Sen's liberal paradox that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism. And we have also shown that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.

In Baryshnikov [3] he said, "the similarities between the two theories, the classical and topological ones, are somewhat more extended than one would expect. The details seem to fit too well to represent just an analogy. I would conjecture that the homological way of proving results in both theories is a 'true' one because of its uniformity and thus can lead to much deeper understanding of the structure of social choice. To understand this structure better we need a much more evolved collection of examples of unifying these two theories and I hope this can and will be done". This paper is an attempt to provide such an example.

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