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On the equivalence of the HEX game theorem and the Duggan–Schwartz theorem for strategy-proof social choice correspondences ☆

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Abstract

Gale [D. Gale, The game of HEX and the Brouwer fixed-point theorem, American Mathematical Monthly 86 (1979) 818–827] has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this paper we will show that under some assumptions about marking rules of HEX games, the HEX game theorem for a 6×6 HEX game is equivalent to the Duggan–Schwartz theorem for strategy-proof social choice correspondences [J. Duggan, T. Schwartz, Strategic manipulability without resoluteness or shared beliefs: Gibbard–Satterthwaite generalized, Social Choice and Welfare 17 (2000) 85–93] that there exists no social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition, residual resoluteness, and has no dictator.

Keywords: The HEX game theorem; Strategy-proof social choice correspondences; The Duggan–Schwartz theorem

1. Introduction

Ref. [5] has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this paper we will show that under some assumptions about marking rules of HEX games, the HEX game theorem for a 6×6 HEX game is equivalent to the Duggan–Schwartz theorem for strategy-proof social choice correspondences [3] that there exists no social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition, residual resoluteness, and has no dictator.[1]

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[1] In another paper, [10], we have shown the equivalence of the HEX game theorem and the Arrow impossibility theorem [1]. The present paper will apply this idea to the problem of the existence of a dictator for strategy-proof social choice correspondences. Also we are proceeding researches about the relationship between the HEX game theorem and other theorems of social choice theory such as the Gibbard–Satterthwaite theorem [6,7] and the strong candidate stability theorem by [4].

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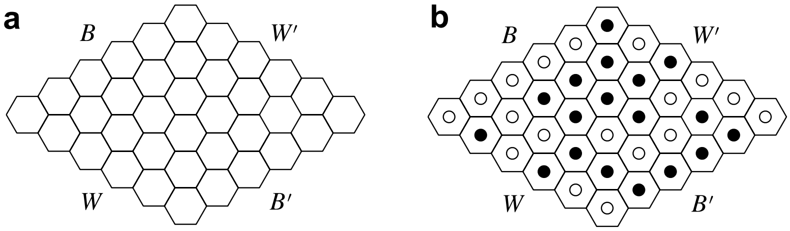

Fig. 1. HEX game.

In the next section according to [5] we present an outline of the HEX game. In Section 3 we will show that the HEX game theorem implies the Duggan–Schwartz theorem. And in Section 4 we will show that the Duggan–Schwartz theorem implies the HEX game theorem.

2. The HEX game

According to Ref. [5] we present an outline of the HEX game. Fig. 1a represents a 6×6 HEX board.[2] Generally a HEX game is represented by an $n \times n$ HEX board where n is a finite positive integer. The rules of the game are as follows. Two players (called Mr. W and Mr. B) move alternately, marking any previously unmarked hexagon or tile with a white (by Mr. W) or a black (by Mr. B) circle respectively. The game has been won by Mr. W (or Mr. B) if he has succeeded in marking a connected set of tiles which meets the boundary regions W and W' (or, B and B'). A set S of tiles is connected if any two members of the set h and h' can be joined by a path $P = (h = h^1, h^2, \ldots, h^m = h')$ where h^i and h^{i+1} are adjacent. Fig. 1b represents a HEX game which has been won by Mr. B.

About the HEX game [5] has shown the following theorem.

Theorem 1 (The HEX game theorem). *If every tile of the HEX board is marked by either a white or a black circle, then there is a path connecting regions W and W', or a path connecting regions B and B'.*

Actually he has shown the theorem that any hex game can never end in a draw, and there always exists at least one winner. But, from his intuitive explanation using the following example of river and dam, it is clear that there exists only one winner in any hex game.

> Imagine that B and B' regions are portions of opposite banks of the river which flow from W region to W' region, and that Mr. B is trying to build a dam by putting down stones. He will have succeeded in damming the river if and only if he has placed his stones in a way which enables him to walk on them from one bank (B region) to the other (B' region).

The proof of Theorem 1 and also the above intuitive argument do not depend on the rule "two players move *alternately*". Therefore, this theorem is valid for any marking rule.

Fig. 2a is obtained by plotting the center of each hexagon, and connecting these centers by lines. Rotating this graph 45° in anticlockwise direction, we obtain Fig. 2b. It is an equivalent representation of the HEX board depicted in Fig. 1a. W and W' represent the regions of Mr. W, and B and B' represent the regions of Mr. B. We call it a square HEX board, and call a game represented by a square HEX board a *square HEX game*. In Fig. 2b we depict an example of winning marking by Mr. B. It corresponds to the marking pattern in Fig. 1b. A set of marked vertices which represents one player's victory is called a winning path.

3. The HEX game theorem implies the Duggan–Schwartz theorem

There are $m(\geqslant 3)$ alternatives and $n(\geqslant 2)$ individuals. m and n are finite positive integers. The set of individuals is denoted by N, the set of alternatives is denoted by A, and the set of all subsets of A is denoted

[2] About the HEX game see also [2].

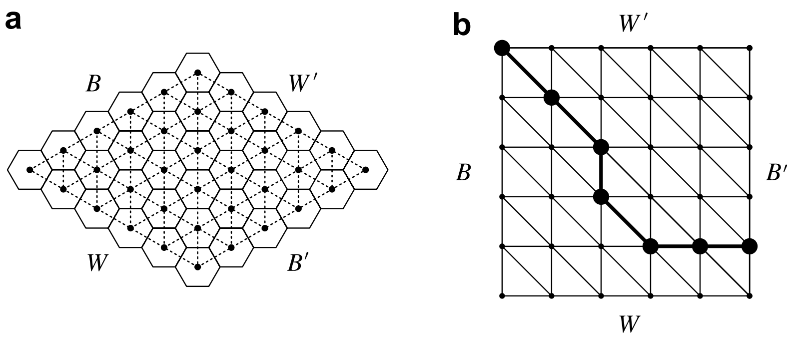

Fig. 2. Square HEX game and winning path.

by $\mathscr{A}$. Denote individual i's preference by P_i. A combination of individual preferences, which is called a *profile*, is denoted by $p(=(P_1, P_2, \ldots, P_n))$, and the set of profiles is denoted by $\mathscr{P}^n$. The alternatives are represented by x, y, z and so on. Individual preferences over the alternatives are strong (or linear) orders, that is, individuals strictly prefer one alternative to another, and are not indifferent about any pair of alternatives.

We consider a social choice correspondence which chooses one or more alternatives corresponding to each profile of *revealed preferences* of the individuals. It is a mapping of $\mathscr{P}^n$ into $\mathscr{A}$. Given profiles $p, p', p'', \ldots$ we denote by $C(p), C(p'), C(p''), \ldots$ the set of alternatives chosen by a social choice correspondence at each profile. We call $C(p)$ the *social choice set* at p, $C(p')$ the social choice set at p', and so on. $C(p)$ at every p is not empty.

We assume the conditions of non-imposition (or citizens' sovereignty) and residual resoluteness. The means of these conditions are as follows.

Non-imposition: For any social choice correspondence and every alternative x there is a profile p at which x is chosen by the social choice correspondence, that is, $x \in C(p)$.

Residual resoluteness: Assume that at a profile p all but one (denoted by i) individuals have the same preferences, and they most prefer an alternative x and secondly prefer another alternative y. And assume that individual i has the same preference as those of other individuals, or only x and y are interchanged in his preference. Then, $C(p)$ is a singleton (the social choice correspondence chooses only one alternative).

As demonstrated by Duggan and Schwartz [3] residual resoluteness is an appropriate condition if the number of individuals is not so small.

Next we consider strategy-proofness according to the definition by Duggan and Schwartz [3]. We assume that each individual (represented by i) has a von Neumann–Morgenstern utility function u_i. If $u_i(x) > u_i(y)$ when individual i prefers x to y for arbitrary pair of alternatives (x, y), the preference of individual i *is represented by* u_i.

Let p and p' be two profiles between which only the preference of individual i differs. $C(p)$ and $C(p')$ are the social choice sets at p and p'. Assume that individual i assigns probabilities $\mu(x)$ and $\mu'(x)$ to an alternative x included in $C(p)$ and $C(p')$, and so on. $\mu(x)$ is individual i's subjective probability of alternative x when $C(p)$ is the social choice set, and $\mu'(x)$ has similar meaning. Then, his expected utilities at p and p' evaluated by his utility function at p are

$$E_i(p) = \sum_{x \in C(p)} \mu(x) u_i(x) \quad \left(\text{where} \sum_{x \in C(p)} \mu(x) = 1\right)$$

and

$$E_i(p') = \sum_{x \in C(p')} \mu'(x) u_i(x) \quad \left(\text{where} \sum_{x \in C(p')} \mu'(x) = 1\right).$$

If for *all* assignments of probabilities to alternatives we have

$$E_i(p') > E_i(p), \tag{1}$$

then individual i has an incentive to report his preference P'_i when his true preference is P_i, and the social choice correspondence is *manipulable* by him at p. Conversely, if for *some* assignment of probabilities we have $E_i(p) \geqq E_i(p')$, then the social choice correspondence is not manipulable.

Now we can show the following lemma.

Lemma 1. *Let p and p' be two profiles of individual preferences between which only the preference of one individual (denoted by i) differs. If and only if for some $x \in C(p')$ and all $y \in C(p)$, or for some $y \in C(p)$ and all $x \in C(p')$ individual i prefers x to y at p, the social choice correspondence is manipulable by him at p.*

Proof. First consider the case where for some $x \in C(p')$ and all $y \in C(p)$ individual i prefers x to y at p. Let $\varepsilon > 0$ be the probability of x assigned by him at p', w be his top-ranked (most preferred) alternative in $C(p)$, v be his bottom-ranked (least preferred) alternative in $C(p')$ evaluated by his utility function at p, u_i. Then, we obtain

$$E_i(p') \geqq \varepsilon u_i(x) + (1-\varepsilon)u_i(v)$$

and

$$E_i(p) \leqq u_i(w).$$

Since $u_i(x) > u_i(w)$ and $u_i(x) \geqq u_i(v)$, given ε we can determine the value of $u_i(x)$ so that $E_i(p') > E_i(p)$ holds.

Next consider the case where for some $y \in C(p)$ and all $x \in C(p')$ individual i prefers x to y at p. Let $\varepsilon > 0$ be the probability of y assigned by him at p, w be his bottom-ranked (least preferred) alternative in $C(p')$, and v be his top-ranked (most preferred) alternative in $C(p)$ evaluated by his utility function at p, u_i. Then we obtain

$$E_i(p') \geqq u_i(w)$$

and

$$E_i(p) \leqq \varepsilon u_i(y) + (1-\varepsilon)u_i(v).$$

Since $u_i(y) < u_i(w)$ and $u_i(y) \leqq u_i(v)$, given ε we can determine the value of $u_i(y)$ so that $E_i(p') > E_i(p)$ holds.

Finally, assume that there exists no $x \in C(p')$ such that individual i prefers x to y for all $y \in C(p)$, and no $y \in C(p)$ such that he prefers x to y for all $x \in C(p')$ at p. Let x be his top-ranked alternative in $C(p')$ and y be his bottom-ranked alternative in $C(p)$ evaluated by his utility function at p, u_i. Then, there exists at least one $w \in C(p)$ such that individual i prefers w to x and at least one $z \in C(p')$ such that individual i prefers y to z at p. Let ε' and ε be, respectively, the probability of z at p' and the probability of w at p assigned by him. Then, we obtain

$$E_i(p') \leqq \varepsilon' u_i(z) + (1-\varepsilon')u_i(x)$$

and

$$E_i(p) \geqq \varepsilon u_i(w) + (1-\varepsilon)u_i(y).$$

Since $u_i(w) > u_i(x)$ and $u_i(y) > u_i(z)$, if we assume $\varepsilon = 1 - \varepsilon'$, we obtain $E_i(p) > E_i(p')$, and (1) does not hold. □

Taylor [11] defined that a social choice correspondence is *manipulated by an optimist* in the case where for some $x \in C(\mathbf{p}')$ and all $y \in C(\mathbf{p})$ an individual prefers x to y at $\mathbf{p}$, and defined that it is *manipulated by a pessimist* in the case where for some $y \in C(\mathbf{p})$ and all $x \in C(\mathbf{p}')$ an individual prefers x to y at $\mathbf{p}$.

Strategy-proofness is defined as follows:

Strategy-proofness: If a social choice correspondence is not manipulable for any individual at any profile, it is *strategy-proof*.

Further we define some terminologies as follows.

Monotonicity: Let $C(p)$ be the social choice set at some profile p, y be an alternative outside $C(p)$, and assume the following individual preferences at p.

(1) Individuals in a group S ($S \subseteq N$) prefer x to y for some $x \in C(p)$.
(2) Others (group $S' = N - S$) prefer y to x for all $x \in C(p)$.

Consider another profile $p' \in \mathscr{P}^n$ such that individuals in S are partitioned into the following l sub-groups.[3]

(1) S_1: For some set of alternatives X_1 which includes $C(\mathbf{p})$ and does not include y ($C(\mathbf{p}) \subset X_1$ and $y \notin X_1$), individuals in S_1 prefer x' to z for all $x' \in X_1$ and all $z \notin X_1$ at $\mathbf{p}'$.
(2) S_2: For some set of alternatives X_2 which includes X_1 and does not include y ($X_1 \subset X_2$ and $y \notin X_2$), individuals in S_2 prefer x' to z for all $x' \in X_2$ and all $z \notin X_2$ at $\mathbf{p}'$.
(3) $\cdots$
(4) S_{l-1}: For some set of alternatives X_{l-1} which includes X_{l-2} and does not include y ($X_{l-2} \subset X_{l-1}$ and $y \notin X_{l-1}$), individuals in S_{l-1} prefer x' to z for all $x' \in X_{l-1}$ and all $z \notin X_{l-1}$ at $\mathbf{p}'$.
(5) S_l: Their preferences do not change.

The preferences of individuals in S' at p' are not specified. Then, the social choice correspondence does not choose y at p' ($y \notin C(p')$).

Semi-decisive: A group of individuals S is *semi-decisive* for x against y if when for some set of alternatives X such that $x \in X$ and $y \notin X$ individuals in S prefer x' to z for all $x' \in X$ and all $z \notin X$, a social choice correspondence does not choose y regardless of the preferences of other individuals.

Semi-decisive set: If S is semi-decisive about all pairs of alternatives, it is called a semi-decisive set.

S in the definition of semi-decisive set may consists of one individual. If, for a social choice correspondence, a set of one individual is a semi-decisive set, then this individual is a dictator of the social choice correspondence because any alternative other than his most preferred alternative is never chosen.

Now we show the following result.

Lemma 2. *If a social choice correspondence is strategy-proof, then it satisfies monotonicity.*

In the following proof we use notations in the definition of monotonicity, and we neglect individuals in S_l whose preferences do not change between p and p'.

Proof. Without loss of generality let individuals 1 to m ($0 \leqq m \leqq n$) belong to S and individuals $m+1$ to n belong to S'. Consider a profile p'' other than p and p' such that individuals in S prefer x to y to z for all $x \in C(p)$, and individuals in S' prefer y to x to z for all $x \in C(p)$, where z is an arbitrary alternative other than alternatives in $C(p)$ and y.

Let p^1 be a profile such that only the preference of individual 1 changes from P_1 (his preference at p) to P''_1 (his preference at p''), and suppose that at p^1 an alternative other than alternatives in $C(p)$ is included in the social choice set. Then, he has an incentive to report a false preference P_1 when his true preference is P''_1 because he prefers alternatives in $C(p)$ to all other alternatives at p^1. Therefore, at p^1 only alternatives in $C(p)$ are chosen by the social choice correspondence. By the same logic, when the preferences of individual 1 to m change from P_i to P''_i, only alternatives in $C(p)$ are chosen. Next, let p^{m+1} be a profile such that the preference of individual $m+1$, as well as the preferences of the first m individuals, changes from P_{m+1} to P''_{m+1}, and suppose that at p^{m+1} y is included in the social choice set. Then, individual $m+1$ has an incentive to report a false preference P''_{m+1} when his true preference is P_{m+1} because at p he prefers y to x for all $x \in C(p)$. On the other hand, if an alternative other than alternatives in $C(p)$ is included in the social choice set at p^{m+1}, he has an incentive to report a false preference P_{m+1} when his true preference is P''_{m+1} because at p^{m+1} he prefers x to z for all $x \in C(p)$ and all $z \notin C(p), z \neq y$. By the same logic, when the preferences of all individuals change from P_i to P''_i, only alternatives in $C(p)$ are chosen by the social choice correspondence.

Now, suppose that from p'' to p' the individual preferences change one by one from P''_i to P'_i. If, when the preference of the first individual in S_1 changes, an alternative outside X_1 is chosen, he has an incentive to

[3] The number of sub-groups l does not exceed the number of individuals who belong to S.

report a false preference P''_i when his true preference is P'_i because at p' he prefers x to z for all $x \in X_1$ and all $z \notin X_1$. Consequently only some alternatives included in X_1 are chosen. By the same logic, when the preferences of all individuals in S_1 change from P''_i to P'_i, only some alternatives in X_1 are chosen. Similarly, when the preferences of all individuals in S_2 (denoted by j) change from P''_j to P'_j, only some alternatives in X_2 are chosen, ..., when the preferences of all individuals in S_{l-1} (denoted by k) change from P''_k to P'_k, only some alternatives in X_{l-1} are chosen. Further, if, when the preference of the first individual (individual $m+1$) in S' changes, y is included in the social choice set, then he has an incentive to report a false preference P'_{m+1} when his true preference is P''_{m+1} because at p'' he prefers y to z for all $z \neq y$. By the same logic, when the preferences of all individuals change, y is not chosen by the social choice correspondence. □

The Duggan–Schwartz theorem states that there exists a dictator for any strategy-proof social choice correspondence which satisfies the conditions of non-imposition and residual resoluteness, or in other words, there exists no social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition, residual resoluteness, and has no dictator. A dictator for a social choice correspondence is an individual such that the social choice correspondence always chooses only his most preferred alternative, or in other words the social choice set always includes only his most preferred alternative.

About the concepts of semi-decisiveness and semi-decisive set we will show some results. As preliminary results we show the following lemmas.

Lemma 3 (Unanimity). *Suppose that a social choice correspondence satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness. If at a profile p all individuals most prefer an alternative (denoted by x), then the social choice correspondence chooses only this alternative, that is,* $C(p) = \{x\}$.

Proof. Consider a profile p' at which all individuals have the same preferences and they most prefer x. By residual resoluteness only one alternative is chosen by the social choice correspondence. By non-imposition at some profile p'' x is chosen ($x \in C(p'')$). If, when the preference of one individual (individual 1) changes from P''_1 (his preference at p'') to P'_1 (his preference at p'), x is not chosen by the social choice correspondence, then individual 1 has an incentive to report a false preference P''_1 when his true preference is P'_1 because he most prefers x at p', and the social choice correspondence is manipulable by individual 1. Thus, x is chosen in this case. By the same logic x is chosen at p'. By residual resoluteness at p' only x is chosen ($C(p') = \{x\}$).

Next, if, when the preference of one individual (individual 1) changes from P'_1 to P_1 (his preference at p), an alternative other than x is chosen by the social choice correspondence, then he has an incentive to report a false preference P'_1 when his true preference is P_1 because he most prefers x at p, and the social choice correspondence is manipulable by individual 1. By the same logic any alternative other than x is not chosen at p, and we have $C(p) = \{x\}$. □

Lemma 4. *Suppose that a social choice correspondence satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness.*

(1) *Let partition the individuals into the following two groups, and for alternatives x, y and w we assume the following profile p:*
  (i) *Individuals in a group S:* $xP_iyP_iwP_iz$.
  (ii) *Others (N/S):* $yP_iwP_ixP_iz$.
  where z denotes an arbitrary alternative other than x, y and w. Then, the social choice correspondence does not choose any alternative other than x and y.
(2) *Similarly, let partition the individuals into the following two groups, and for alternatives x, y and w we assume the following profile p:*
  (i) *Individuals in S:* $wP_ixP_iyP_iz$.
  (ii) *Others (N/S):* $yP_iwP_ixP_iz$.
  where z denotes an arbitrary alternative other than x, y and w. Then, the social choice correspondence does not choose any alternative other than y and w.

Proof

(1) By Lemma 3 there is a profile p' at which $C(p') = \{y\}$. Suppose that, starting from individuals outside S, their preferences change from P_i' to P_i (from profile p' to p) one by one. Even when the preferences of individuals outside S change, only y is chosen because they most prefer y at p. On the other hand, when the preferences of individuals in S change, any alternative other than x and y is not chosen because they most prefer x and secondly prefer y at p.

(2) Permuting w, x and y and interchanging S and N/S, the proof of this case is the same as the proof of (1). □

Next we show.

Lemma 5. *Suppose that a social choice correspondence satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness. If a group S is semi-decisive about one pair of alternatives, then it is a semi-decisive set.*

Proof. Assume that S is semi-decisive for x against y. Let w be an alternative other than x and y.

(1) Consider the following profile p.
  (i) Individuals in S prefer x to y to w to z.
  (ii) Other individuals prefer y to w to x to z.
  z denotes an arbitrary alternative other than x, y and w. Since S is semi-decisive for x against y we have $y \notin C(p)$. From Lemma 4 we have $w \notin C(p)$ and $z \notin C(p)$, and so we have $C(p) = \{x\}$. Individuals in S prefer x to w, but all other individuals prefer w to x. Therefore, by monotonicity S is semi-decisive for x against w.

(2) Next consider the following profile p':
  (i) Individuals in S prefer w to x to y to z.
  (ii) Other individuals prefer y to w to x to z.
  z denotes an arbitrary alternative other than x, y and w. Since S is semi-decisive for x against y we have $y \notin C(p)$. From Lemma 4 we have $x \notin C(p)$ and $z \notin C(p)$, and so we have $C(p) = w$. Individuals in S prefer w to y, but all other individuals prefer y to w. Therefore, by monotonicity S is semi-decisive for w against y.

Applying this logic repeatedly we can show that S is a semi-decisive set. □

The implications of this lemma are similar to those of Lemma 3*a in [8] and Dictator Lemma in [9] for binary social choice rules.

Now we confine us to a subset of profiles $\overline{\mathscr{P}}^n$ such that all individuals prefer three alternatives x, y and z to all other alternatives. Unanimity implies that the set of all individuals N is semi-decisive about every pair of alternatives, and so it is a semi-decisive set. Thus, by monotonicity any social choice correspondence does not choose any alternative other than x, y and z at all such profiles. We denote individual preferences about x, y and z in this subset of profiles as follows:

$$p^1 = (123),\quad p^2 = (132),\quad p^3 = (312),\quad p^4 = (321),\quad p^5 = (231),\quad p^6 = (213),$$

$p^1 = (123)$ represents all preferences such that an individual prefers x to y to z to all other alternatives, $p^1 = (132)$ represents all preferences such that an individual prefers x to z to y to all other alternatives, and so on. Although we confine our arguments to such a subset of profiles, Lemma 5 with monotonicity ensures that an individual who is semi-decisive about a pair of alternatives for this subset of profiles is a dictator for all profiles.

From Lemma 5 for the profiles in $\overline{\mathscr{P}}^n$ we obtain the following result.

Lemma 6. *If two groups S and S′, which are not disjoint, are semi-decisive sets, then their intersection $S \cap S'$ is a semi-decisive set.*

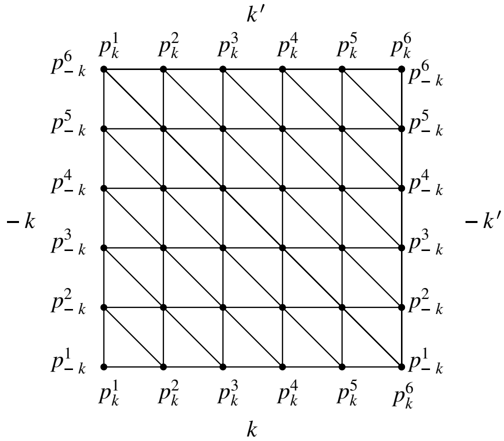

Fig. 3. HEX game representing profiles.

Proof. For three alternatives x, y and z we consider the following profile:

(1) Individuals in $S\backslash(S\cap S')$ prefer z to x to y.
(2) Individuals in $S'\backslash(S\cap S')$ prefer y to z to x.
(3) Individuals in $S\cap S'$ prefer x to y to z.
(4) Individuals in $N\backslash(S\cup S')$ prefer z to y to x.

Since S and S' are semi-decisive sets, by monotonicity the social choice correspondence does not choose y and z. Thus, it chooses x. Since only individuals in $S\cap S'$ prefer x to z and all other individuals prefer z to x, by monotonicity $S\cap S'$ is semi-decisive for x against z. From Lemma 5 it is a semi-decisive set. □

Further we confine us to a subset of $\overline{\mathscr{P}}^n$ such that all but one individual have the same preferences, and consider a HEX game between one individual (denoted by individual k) and the set of individuals other than k. Representative profiles are denoted by $(p^i_k, p^j_{-k}),\ i=1,\ldots,6,\ j=1,\ldots,6$, where p^i_k is individual k's preference and p^j_{-k} denotes the common preference of individuals other than k. We relate these profiles to the vertices of a 6×6 square HEX board as depicted in Fig. 3. There are 36 vertices in this HEX board. It represents a square HEX game. k and k' represent individual k's regions, and $-k$ and $-k'$ represent the regions of the set of individuals other than k.

We consider the following marking and winning rules of square HEX games.

(1) At a profile represented by a vertex of a square HEX board, if the social choice correspondence chooses *only* the most preferred alternative of individual k which is different from the most preferred alternative of the individuals other than k, then this vertex is marked by a white circle; conversely if the social choice correspondence chooses only the most preferred alternative of the individuals other then k which is different from the most preferred alternative of individual k, then this vertex is marked by a black circle.
(2) A vertex which corresponds to any other profile is randomly marked by a white or a black circle.
(3) The game has been won by individual k (or the set of individuals other than k) if he has (or they have) succeeded in marking a connected set of vertices which meets the boundary regions k and k' (or $-k$ and $-k'$).

A square HEX game is equivalent to the original HEX game. Therefore, there exists one winner for any marking rule. Now we show the following theorem.

Theorem 2. *The HEX game theorem implies the existence of a dictator for any social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness.*

Proof. Since diagonal vertices are not connected, the diagonal path

$$((p_k^1, p_{-k}^1), (p_k^2, p_{-k}^2), \dots, (p_k^6, p_{-k}^6))$$

cannot be a winning path. Consider a profile $(p_k^1, p_{-k}^6) = ((123), (213))$. Unanimity (Lemma 3) and monotonicity mean that z is not chosen by the social choice correspondence at this profile.[4]

By residual resoluteness only one alternative is chosen. Suppose that at this profile the social choice correspondence chooses only y which is the most preferred alternative of the individuals other than k. Then, by monotonicity the social choice correspondence chooses only y at the following profiles:

$$(p_k^1, p_{-k}^5), \quad (p_k^2, p_{-k}^6).$$

[*Note*] At a profile (p_k^1, p_{-k}^1) only x is chosen by unanimity. Then, monotonicity means that z is not chosen at (p_k^2, p_{-k}^6).[5] The fact that at (p_k^1, p_{-k}^6) only y is chosen and monotonicity imply that z is not chosen at (p_k^1, p_{-k}^5), and imply that x is not chosen at (p_k^1, p_{-k}^5) and (p_k^2, p_{-k}^6). We can apply similar arguments to other cases.

The fact that the social choice correspondence chooses only y at a profile (p_k^2, p_{-k}^6) and monotonicity imply that the social choice correspondence chooses only y at the following profiles:

$$(p_k^3, p_{-k}^5), \quad (p_k^4, p_{-k}^5), \quad (p_k^4, p_{-k}^6).$$

Similarly consider a profile $(p_k^3, p_{-k}^2) = ((312), (132))$. By unanimity and monotonicity y is not chosen by the social choice correspondence. By residual resoluteness only one alternative is chosen. Suppose that at this profile the social choice correspondence chooses only x which is the most preferred alternative of the individuals other than k. Then, by monotonicity the social choice correspondence chooses only x at the following profiles:

$$(p_k^3, p_{-k}^1), \quad (p_k^4, p_{-k}^2).$$

The fact that the social choice correspondence chooses only x at a profile (p_k^4, p_{-k}^2) and monotonicity imply that the social choice correspondence chooses only x at the following profiles:

$$(p_k^5, p_{-k}^1), \quad (p_k^6, p_{-k}^1), \quad (p_k^6, p_{-k}^2).$$

Similarly consider a profile $(p_k^5, p_{-k}^4) = ((231), (321))$. By unanimity and monotonicity x is not chosen by the social choice correspondence. By residual resoluteness only one alternative is chosen. Suppose that at this profile the social choice correspondence chooses only z which is the most preferred alternative of the individuals other than k. Then, by monotonicity the social choice correspondence chooses only z at the following profiles:

$$(p_k^5, p_{-k}^3), \quad (p_k^6, p_{-k}^4).$$

The fact that the social choice correspondence chooses only z at a profile (p_k^6, p_{-k}^4) and monotonicity imply that the social choice correspondence chooses only z at the following profiles:

$$(p_k^1, p_{-k}^3), \quad (p_k^2, p_{-k}^3), \quad (p_k^2, p_{-k}^4).$$

The vertices which correspond to all of these profiles are marked by black circles. Then, even when all other vertices are marked by white circles, we obtain a marking pattern of a square HEX board as depicted in Fig. 4. The set of individuals other than k is the winner of this game. Therefore, for individual k to be the winner of the square HEX game, the alternative chosen by the social choice correspondence must coincide with the most preferred alternative of individual k at least at one of three profiles (p_k^1, p_{-k}^6), (p_k^3, p_{-k}^2) and (p_k^5, p_{-k}^4). Then, by monotonicity individual k is semi-decisive about at least one pair of alternatives, and then by Lemma 5 he is a dictator.

If for all k $(k = 1, 2, \dots, n)$, individual k is not the winner of all square HEX games between individual k and the set of individuals other than k, then each set of individuals excluding one individual is the winner of each square HEX game. By Lemma 6 every non-empty intersection of the sets of individuals excluding one individual is a semi-decisive set. Then, the intersection of $N\setminus 1, N\setminus\{2\}, \dots, N\setminus\{n-1\}$ is a semi-decisive set. But $(N\setminus\{1\}) \cap (N\setminus\{2\}) \cap \dots \cap (N\setminus\{n-1\}) = \{n\}$. Thus, individual n is a dictator. Therefore, the HEX game theorem implies the existence of a dictator for any social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness. □

By this theorem the HEX game theorem implies the Duggan–Schwartz theorem.

[4] Here X_1 in the definition of monotonicity is $\{x, y\}$.

[5] Here X_1 and X_2 in the definition of monotonicity are x and x, y.

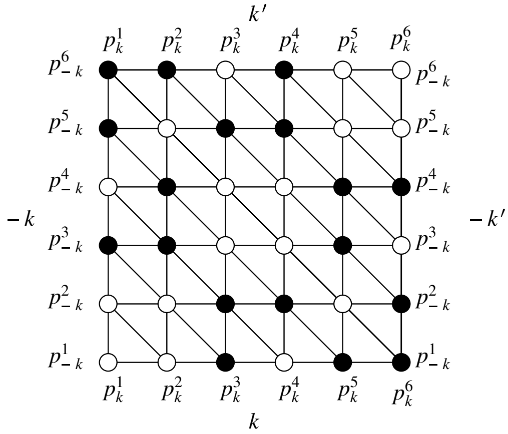

Fig. 4. Winning path of a square HEX game.

4. The Duggan–Schwartz theorem implies the HEX game theorem

Next we show that the Duggan–Schwartz theorem implies the HEX game theorem under an interpretation of dictator. Similarly to the previous section, we confine us to a subset of profiles such that all individuals prefer three alternatives x, y and z to all other alternatives, and the preferences of individuals other than one individual (denoted by k) are the same. And we consider a square HEX game between individual k and the set of individuals other than k. The dictator of a social choice correspondence is interpreted as an individual who can determine the choice of the society when his most preferred alternative and that of the other individuals are different, and in a HEX game he can mark tiles with his color in such cases. Without loss of generality we assume that individual k is a dictator of a social choice correspondence. We denote a vertex of a square HEX board which corresponds to a profile (p_k^i, p_{-k}^j) simply by (p_k^i, p_{-k}^j). If individual k is a dictator, the following vertices are marked be white circles:

$$(p_k^1, p_{-k}^3),\quad (p_k^1, p_{-k}^4),\quad (p_k^1, p_{-k}^5),\quad (p_k^1, p_{-k}^6),\quad (p_k^2, p_{-k}^3),\quad (p_k^2, p_{-k}^4),\quad (p_k^2, p_{-k}^5),\quad (p_k^2, p_{-k}^6),$$
$$(p_k^3, p_{-k}^1),\quad (p_k^3, p_{-k}^2),\quad (p_k^3, p_{-k}^5),\quad (p_k^3, p_{-k}^6),\quad (p_k^4, p_{-k}^1),\quad (p_k^4, p_{-k}^2),\quad (p_k^4, p_{-k}^5),\quad (p_k^4, p_{-k}^6),$$
$$(p_k^5, p_{-k}^1),\quad (p_k^5, p_{-k}^2),\quad (p_k^5, p_{-k}^3),\quad (p_k^5, p_{-k}^4),\quad (p_k^6, p_{-k}^1),\quad (p_k^6, p_{-k}^2),\quad (p_k^6, p_{-k}^3),\quad (p_k^6, p_{-k}^4).$$

Then, we obtain Fig. 5. Unmarked vertices, where the most preferred alternatives of all individuals are the same, should be randomly marked. Clearly individual k is the winner of this HEX game. Thus, the existence

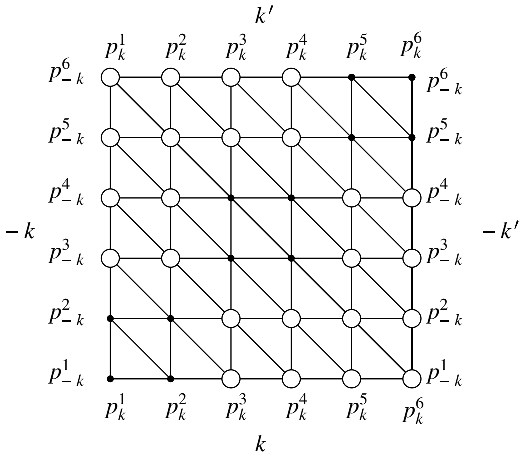

Fig. 5. HEX game won by individual k.

of a dictator for a social choice correspondence implies the existence of a winner for a HEX game. Therefore, we obtain.

Theorem 3. *The Duggan–Schwartz theorem and the HEX game theorem are equivalent.*

5. Concluding remarks

We have considered the relationship between the HEX game theorem and the Duggan–Schwartz theorem, and have shown their equivalence. We think that the idea of this paper can be applied to other social choice theorems which argue the existence of a dictator for some social choice rules.

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