A topological proof of Eliaz’s unified theorem of social choice theory

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Abstract

Recently Eliaz [Social aggregators, Social Choice and Welfare 22 (2004) 317–330.] has presented a unified framework to study (Arrovian) social welfare functions and non-binary social choice functions based on the concept of preference reversal. He showed that social choice rules which satisfy the property of preference reversal and a variant of the Pareto principle are dictatorial. This result includes the Arrow impossibility theorem [Arrow, Social Choice and Individual Values, Second ed., Yale University Press, 1963.] and the Gibbard–Satterthwaite theorem [Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973) 587–601; Satterthwaite, Strategy-proofness and Arrow’s conditions: existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory 10 (1975) 187–217.] as its special cases. We present a concise proof of his theorem using elementary concepts of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings.

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1. Introduction

Recently Eliaz [8] has presented a unified framework to study (Arrovian) social welfare functions and non-binary social choice functions based on the concept of preference reversal. The preference reversal property is a condition (according to the expression in Eliaz [8]) that if social relation (given by a social choice function or a social preference) between any two alternatives has been reversed, then someone must have exhibited the same reversal in his preference. He showed that social choice rules which satisfy the property of preference reversal and a variant of the Pareto principle are dictatorial. This result includes the Arrow impossibility theorem [1] and the Gibbard–Satterthwaite theorem [9,15] as its special cases. We present a concise proof of his theorem using elementary concepts of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings.

Topological approaches to social choice problems have been initiated by Chichilnisky [6]. Her main result is an impossibility theorem that there exists no continuous social choice rule which satisfies unanimity and anonymity. This approach has been further developed by Chichilnisky, Candeal and Indurain, Koshevoy,
Lauwers, Weinberger [5,7,4,11,13,17], and so on. On the other hand, Baryshnikov [2,3] have presented a topological approach to the Arrow impossibility theorem (or general possibility theorem) in a discrete framework of social choice.1 Our research is in line with the studies of topological approaches to discrete social choice problems initiated by him. In the following section, we present expressions of binary social choice rules by simplicial complexes and simplicial mappings. In Section 3, we will prove the main results of this paper.

2. The model

There are \( m \) alternatives of a social problem, \( x_1, x_2, \ldots, x_m (m \geq 3) \), and \( n \) individuals \( (n \geq 2) \). The set of alternatives is denoted by \( A \). \( m \) and \( n \) are finite integers. Individual preferences over these alternatives are complete, transitive and asymmetric. Individual \( i \)’s preference is denoted by \( p_i \). \( x_iP_jx_j \) means that he prefers \( x_i \) to \( x_j \).

A social choice rule which we will consider according to Eliaz [8] is a rule that determines a social binary relation about each pair of alternatives corresponding to a combination of individual preferences. It may not be complete. We call such a social choice rule a binary social choice rule. It is abbreviated as BCR. We assume the universal (or unrestricted) domain condition for social binary choice rules.2 We call a combination of individual preferences a profile. The profiles are denoted by \( p \), \( p' \) and so on. Individual \( i \)’s preference at \( p' \) is denoted by \( p'_i \), and so on. A social binary relation generated by a BCR is denoted by \( R \). We call it also a BCR. Let \( x_i \) and \( x_j \) be two distinct alternatives. \( x_iRx_j \) means that \( x_i \) relates to \( x_j \) according to BCR \( R \). On the other hand \( x_iR'x_j \) means that \( x_i \) does not relate to \( x_j \) according to BCR \( R \). A BCR at a profile \( p \) is denoted by \( R \), a BCR at \( p' \) is denoted by \( R' \), and so on.

Any BCR \( R \) is required to satisfy the following conditions.

Existence of a best alternative (BA). There exists an alternative \( x_i \in A \) such that \( x_iRx_j \) for all \( x_j \in A \setminus \{x_i\} \). There may be multiple best alternatives.

Acyclicity (AC). For every three alternatives \( x_i, x_j \) and \( x_k \) in \( A \) if \( x_iRx_j \) and \( x_kRx_j \), then \( x_jRx_i \).

Pareto efficiency (PAR). For every two alternatives \( x_i \) and \( x_j \) in \( A \) if all individuals prefer \( x_i \) to \( x_j \), then either \( "x_iRx_j \) and \( x_jRx_i" \), or \( "x_i \) and \( x_j \) are not related according to \( R (x_iR'x_j \) and \( x_jR'x_i)" \).

Preference reversal (PR). For every two alternatives \( x_i \) and \( x_j \) in \( A \) if \( x_iRx_j \), \( x_jRx_i \) but \( x_iR'x_j \), then there exists (at least) one individual \( i \) such that \( x_iP'_ix_j \) and \( x_iP'_ix_j \).

Dictator is defined as follows.

Dictator. If, there exists an individual \( i \) such that for every pair of alternatives \( x_i \) and \( x_j \) the social relation is \( x_jRx_i \) whenever he prefers \( x_i \) to \( x_j \), then he is the dictator of \( R \).

As proved in Observation 1 of Eliaz [8] AC is equivalent to the following Transitivity.

Transitivity (T). For every three alternatives \( x_p, x_j \) and \( x_k \) in \( A \) if \( x_jRx_k \) and \( x_kRx_p \), then \( x_jRx_p \).

Proof

(1) AC \( \rightarrow \) T: Assume that \( x_jRx_k \), \( x_jRx_p \) but \( x_j'Rx_p \). Then, from \( x_jRx_p \) and \( x_j'Rx_p \) AC implies \( x_j'Rx_j \). It is a contradiction.

(2) T \( \rightarrow \) AC: Assume that \( x_jRx_p \), \( x_jRx_k \) but \( x_jRx_p \). Then, from \( x_kRx_j \) and \( x_jRx_j \) T implies \( x_kRx_p \). It is a contradiction. \( \square \)

As noted by Eliaz [8] if a BCR satisfies BA, AC and the Completeness (Condition C) (\( x_iRx_j \) or \( x_iRx_j \)), then it is an Arrovian social welfare function. In this interpretation AC means the transitivity of strict social preferences.3 Eliaz [8] showed that if a social welfare function satisfies BA, AC, PAR, C and Arrow’s condition of independence of irrelevant alternatives, then it satisfies PR. If a BCR satisfies C, \( x_jRx_j \) is equivalent to \( x_iRx_j \) Thus, the dictator in the above definition is the dictator for an Arrovian social welfare function.

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1 About surveys and basic results of topological social choice theories, see Mehta [14] and Lauwers [12].
2 The universal domain condition means that the domain of individuals preferences for social binary choice rules is never restricted.
3 From Lemma 1 of Baryshnikov [2], we know that if individual preferences are strict orders, then the social preference is also a strict order under the transitivity, the Pareto principle and the independence of irrelevant alternatives.
On the other hand, if the unique alternative \( x_i \) satisfies \( x_i R x_j \) for all \( x_j \in A \setminus \{x_i\} \) and all alternatives other than \( x_i \) are not mutually related according to a BCR \( R \), then it is a social choice function which is a social choice rule that chooses one alternative corresponding to each profile. To be precise a social choice function chooses one alternative corresponding to a profile of reported preferences of individuals. If a social choice function does not give any incentive to every individual to report a preference which is different from his true preference, then it is strategy-proof. It was shown by Eliaz [8] that a strategy-proof social choice function satisfies PR. If there exists the unique best alternative \( x_i \) for a BCR, then \( x_i R x_i \) means that \( x_j \) is not chosen by the social choice function derived from this BCR, and the dictator in the above definition is the dictator for the social choice function. Eliaz [8] showed the theorem that if a BCR satisfies BA, AC, PAR and PR, it has the dictator. Then, the Arrow impossibility theorem that there exists the dictator for any social welfare function which satisfies BA, AC, C, PAR and the independence of irrelevant alternatives under the universal domain condition, and the Gibbard–Satterthwaite theorem that there exists the dictator for any social choice function which is onto (surjection) and strategy-proof under the universal domain condition are the special cases of his theorem.

PAR with BA implies the following condition.4

Strong Pareto efficiency (SPAR) For every alternative \( x_i \) if all individuals prefer \( x_i \) to all other alternatives, then we have \( x_i R x_j \) and \( x_j R x_i \) for all \( x_j \in A \setminus \{x_i\} \).

Now we consider topological expressions of individual preferences. We draw a circumference which represents the set of individual preferences by connecting \( m! \) vertices \( v_1, v_2, \ldots, v_{m!} \) by arcs.5 For example, in the case of four alternatives, these vertices mean the following preferences.

\[
\begin{align*}
  &v_1 : (1234),
  &v_2 : (1243),
  &v_3 : (1423),
  &v_4 : (1432),
  &v_5 : (1342),
  &v_6 : (1324) \\
  &v_7 : (2134),
  &v_8 : (2143),
  &v_9 : (2413),
  &v_{10} : (2431),
  &v_{11} : (2341),
  &v_{12} : (2314) \\
  &v_{13} : (3124),
  &v_{14} : (3142),
  &v_{15} : (3412),
  &v_{16} : (3421),
  &v_{17} : (3241),
  &v_{18} : (3214) \\
  &v_{19} : (4123),
  &v_{20} : (4132),
  &v_{21} : (4312),
  &v_{22} : (4321),
  &v_{23} : (4231),
  &v_{24} : (4213)
\end{align*}
\]

We denote a preference such that an individual prefers \( x_1 \) to \( x_2 \) to \( x_3 \) to \( x_4 \) by (1234), and so on. Notations for the cases with different number of alternatives are similar. Generally \( v_1 \sim v_{(m-1)!} \) represent preferences such that the most preferred alternative for an individual is \( x_1 \), \( v_{(m-1)!+1} \sim v_{2(m-1)!} \) represent preferences such that the most preferred alternative for an individual is \( x_2 \), and so on. In particular \( v_1 \) denotes a preference such that an individual prefers \( x_1 \) to \( x_2 \) to \( x_3 \) to \( \ldots \) to \( x_m \). It is denoted by \((123\ldots m)\).

Denote this circumference by \( S_1^1 \). In the case of three alternatives is depicted in Fig. 1. The set of profiles of the preferences of \( n \) individuals is represented by the product space \( S_1^1 \times \cdots \times S_1^1 \) \((n \text{ times})\). It is denoted by \( (S_1^1)^n \). The one-dimensional homology group of \( S_1^1 \) is isomorphic to the group of integers \( \mathbb{Z} \), that is, \( H_1(S_1^1) \cong \mathbb{Z} \). And the one-dimensional homology group of \( (S_1^1)^n \) is isomorphic to the direct product of \( n \) groups of integers \( \mathbb{Z}^n \), that is, we have \( H_1((S_1^1)^n) \cong \mathbb{Z}^n \). It is proved, for example, using the Mayer–Vietoris exact sequences.6

The social binary relation generated by a BCR is also represented by a circumference depicted in Fig. 2. This circumference is drawn by connecting three vertices, \( w_1, w_2 \) and \( w_3 \) by arcs. These vertices mean the following social binary relations.

1. \( w_2 \): binary relations such that \( x_2 R x_j \) and \( x_j R x_2 \) for all \( x_j \in A \setminus \{x_2\} \).
2. \( w_3 \): binary relations such that \( x_3 R x_j \) and \( x_j R x_3 \) for all \( x_j \in A \setminus \{x_3\} \).
3. \( w_1 \): all other social binary relations.

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4 This term SPAR is not defined in Eliaz [8].
5 \( m! \) denotes factorial of \( m \).
6 About homology groups and the Mayer–Vietoris exact sequences we referred to Tamura [16] and Komiya [10].
We call this circumference $S^1$. The one-dimensional homology group of $S^1$ is also isomorphic to $\mathbb{Z}$, that is, $H_1(S^1) \cong \mathbb{Z}$.

**Binary social choice rules are simplicial mappings.** Binary social choice rules are denoted by $f : (S^1)^n \to S^1$. Two adjacent vertices of $S^1$ span a one-dimensional simplex. And any pair of two vertices of $S^1$ spans a one-dimensional simplex. Thus, $f$ is a simplicial mapping, and we can define the homomorphism of homology groups induced by $f$.

We define an inclusion mapping from $S^1_i$ to $(S^1)^n$ by $\Delta : S^1_i \to (S^1)^n$ under the assumption that all individuals have the same preferences, and define an inclusion mapping when the profile of preferences of individuals other than one individual (denoted by $i$) is fixed at some profile by $i_i : S^1_i \to (S^1)^n$. The homomorphisms of homology groups induced by these inclusion mappings are as follows:

\[
\Delta : \mathbb{Z} \to \mathbb{Z}^n : h \to (h, h, \ldots, h), \quad h \in \mathbb{Z},
\]

\[
i_i : \mathbb{Z} \to \mathbb{Z}^n : h \to (0, \ldots, 0, h, 0, \ldots, 0), \quad h \in \mathbb{Z} \text{ (only the } i\text{th component is } h).
\]

From these definitions we obtain the following relation about $\Delta$ and $i_i$ at any profile.

\[
\Delta = \sum_{i=1}^n i_i.
\] (1)

Let us denote the homomorphism of homology groups induced by $f$ by $f : (\mathbb{Z})^n \to \mathbb{Z}$.

**Binary social choice rules for different profiles are homotopic.** $f$ for a fixed profile of preferences of individuals other than $i$ (denoted by $f|_{p_{-i}}$) and $f$ for another fixed profile of their preferences (denoted by $f|_{p'_{-i}}$) are homotopic. Thus, the homomorphisms of homology groups induced by them are isomorphic. Denote two profiles of individuals other than $i$ by $p_{-i}$ and $p'_{-i}$. Then, the homotopy between $f|_{p_{-i}}$ and $f|_{p'_{-i}}$ is

\[
f_t = \frac{tf|_{p_{-i}} + (1-t)f|_{p'_{-i}}}{|tf|_{p_{-i}} + (1-t)f|_{p'_{-i}}|} \quad (0 \leq t \leq 1).
\]

It is well defined since $f|_{p_{-i}}$ and $f|_{p'_{-i}}$ are not anti-podal.
The composite function of \( i \) and \( f \) is denoted by \( f \circ i : S_1^1 \rightarrow S_1^1 \), and its induced homomorphism of homology groups satisfies \( (f \circ i)_* = f_* \circ i_* \), for all \( i \). The composite function of \( A \) and \( f \) is denoted by \( f \circ A : S_1^1 \rightarrow S_1^1 \), and its induced homomorphism of homology groups satisfies \( (f \circ A)_* = f_* \circ A_* \). From (1) we obtain

\[
(f \circ A)_* = \sum_{i=1}^{n} (f \circ i)_*.
\]

3. The main results

In this section, we will prove the following theorem by Eliaz [8].

**Theorem 1.** There exists the dictator for any BCR which satisfies BA, AC, PAR and PR.

First we show the following lemma which will be used below.

**Lemma 1.** Suppose that a BCR satisfies BA, AC, PAR and PR, and has no dictator. When the preference of one individual (denoted by \( i \)) is \((234 \ldots m)\), and the preferences of all other individuals are \( v_1 \), then we have

\[
x_i R x_j \quad \text{and} \quad x_j R x_i \quad \text{for all} \quad x_j \in A \setminus \{x_1, x_2\}.
\]

**Proof.** Step 1:

Note that \( v_1 \) represents a preference \((123 \ldots m)\). By PAR we have

\[
x_2 R x_j \quad \text{or} \quad x_j R x_2 \quad \text{and} \quad x_j R x_1 \quad \text{for all} \quad x_j \in A \setminus \{x_1, x_2\}.
\]

By BA there are the following three cases about \( x_1 \) and \( x_2 \):

1. Case 1: \( x_2 R x_1 \) and \( x_1 R x_2 \).
2. Case 2: \( x_1 R x_2 \) and \( x_2 R x_1 \).
3. Case 3: \( x_1 R x_2 \) and \( x_2 R x_1 \).

It will be proved that in Case 1 individual \( i \) is the dictator. In Step 1 we consider this case. By PR we have \( x_i R x_2 \) so long as individual \( i \) prefers \( x_2 \) to \( x_1 \). Then, we say that individual \( i \) is decisive for \( x_2 \) against \( x_1 \). Let \( x_j \) and \( x_k \) (\( x_k \neq x_j \)) be alternatives other than \( x_1 \) and \( x_2 \), and consider the following profile.

1. Individual \( i \) prefers \( x_k \) to \( x_2 \) to \( x_1 \) to \( x_j \) to all other alternatives.
2. Other individuals prefer \( x_1 \) to \( x_j \) to \( x_k \) to all other alternatives.

By PR we have \( x_i R x_2 \). And by PAR we have

1. \( x_1 R x_j \) (or \( x_j R x_1 \)) and \( x_1 R x_i \) (or \( x_j R x_1 \)) and \( x_i R x_1 \) for all \( x_i \in A \setminus \{x_1, x_2, x_j, x_k\} \).
2. \( x_k R x_2 \) (or \( x_2 R x_k \)) and \( x_2 R x_i \) (or \( x_i R x_k \)) and \( x_j R x_k \) for all \( x_i \in A \setminus \{x_1, x_2, x_j, x_k\} \).

BA and AC imply that we have \( x_k R x_j \) for all \( x_j \in A \setminus \{x_k\} \). Then, by PR we have \( x_j R x_k \) so long as individual \( i \) prefers \( x_k \) to \( x_j \), and so individual \( i \) is decisive for \( x_k \) against \( x_j \). Note that \( x_j \) and \( x_k \) are arbitrary. Next consider the following profile.

1. Individual \( i \) prefers \( x_2 \) to \( x_k \) to \( x_j \) to all other alternatives.
2. Other individuals prefer \( x_j \) to \( x_2 \) to \( x_k \) to all other alternatives.

---

7 If \( x_i R x_j \) and \( x_2 R x_1 \), then there exists no best alternative.

8 BA implies \( x_k R x_j \) for all \( x_j \in A \setminus \{x_k\} \), and from AC with \( x_k R x_2 \), \( x_j R x_1 \), \( x_2 R x_k \) and \( x_j R x_k \) (\( x_i \in A \setminus \{x_1, x_2, x_j, x_k\} \)) we have \( x_j R x_k \) for all \( x_j \in A \setminus \{x_k\} \).
By PR we have \( x_i^*Rx_k \). And by PAR we have
\[
x_i^*Rx_k \quad \text{or} \quad x_i^*Rx_k
\]
and \( x_i^*Rx_k \) and \( x_i^*Rx_k \) and \( x_i^*Rx_k \) for all \( x_i \in A \setminus \{x_2, x_j, x_k\} \).

BA and AC imply that we have \( x_i^*Rx_k \) and \( x_i^*Rx_k \) for all \( x_i \in A \setminus \{x_2\} \). Then, by PR we have \( x_i^*Rx_k \) so long as individual \( i \) prefers \( x_2 \) to \( x_j \), and so individual \( i \) is decisive for \( x_2 \) against \( x_j \). Next consider the following profile.

1. Individual \( i \) prefers \( x_k \) to \( x_j \) to \( x_2 \) to all other alternatives.
2. Other individuals prefer \( x_2 \) to \( x_k \) to all other alternatives.

By PR we have \( x_i^*Rx_k \). And by PAR we have
\[
x_i^*Rx_k \quad \text{or} \quad x_i^*Rx_k
\]
and \( x_i^*Rx_k \) and \( x_i^*Rx_k \) and \( x_i^*Rx_k \) for all \( x_i \in A \setminus \{x_2, x_j, x_k\} \).

BA and AC imply that we have \( x_i^*Rx_k \) and \( x_i^*Rx_k \) for all \( x_i \in A \setminus \{x_2\} \). Then, by PR we have \( x_i^*Rx_k \) so long as individual \( i \) prefers \( x_k \) to \( x_2 \), and so individual \( i \) is decisive for \( x_k \) against \( x_2 \). By similar procedures we can show that individual \( i \) is decisive for \( x_1 \) against \( x_j \), and is decisive for \( x_k \) against \( x_1 \). Finally, consider the following profile.

1. Individual \( i \) prefers \( x_1 \) to \( x_k \) to \( x_2 \) to all other alternatives.
2. Other individuals prefer \( x_2 \) to \( x_k \) to all other alternatives.

By PR we have \( x_i^*Rx_k \). And by PAR we have
\[
x_i^*Rx_k \quad \text{or} \quad x_i^*Rx_k
\]
and \( x_i^*Rx_k \) and \( x_i^*Rx_k \) and \( x_i^*Rx_k \) for all \( x_i \in A \setminus \{x_1, x_2, x_k\} \).

BA and AC imply that we have \( x_i^*Rx_k \) and \( x_i^*Rx_k \) for all \( x_i \in A \setminus \{x_1\} \). Then, by PR we have \( x_i^*Rx_k \) so long as individual \( i \) prefers \( x_1 \) to \( x_2 \), and individual \( i \) is decisive for \( x_1 \) against \( x_2 \). Therefore, individual \( i \) is the dictator. 9

Step 2:
Next let us consider Case 2 and 3. From (3) we have \( x_i^*Rx_2 \) for all \( x_j \in A \setminus \{x_1, x_2\} \). Then in both Case 2 and 3, \( x_i^*Rx_2 \) and AC imply
\[
x_i^*Rx_2 \quad \text{for all} \quad x_j \in A \setminus \{x_1, x_2\}.
\]

By BA in Case 2 we obtain
\[
x_i^*Rx_2 \quad \text{and} \quad x_i^*Rx_1 \quad \text{for all} \quad x_j \in A \setminus \{x_1\}.
\]

And in Case 3 we have 10
\[
x_i^*Rx_2, x_j^*Rx_1, x_j^*Rx_j \quad \text{and} \quad x_j^*Rx_1 \quad \text{for all} \quad x_j \in A \setminus \{x_1, x_2\}.
\]

Therefore, we get the conclusion of this lemma. \( \square \)

By SPAR we obtain the correspondences from the vertices of \( S^1 \) to the vertices of \( S^1 \) by \( f_0A \) as follows.
\[
v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1 \quad \text{or} \quad v_1 \sim v_1.
\]

All other vertices correspond to \( w_i \). Sets of one-dimensional simplices included in \( S^1 \) which are one-dimensional cycles are only the following \( z \) and its counterpart \( -z \).
\[
z = \{v_1, v_2\} + \{v_2, v_3\} + \cdots + \{v_{m-1}, v_m\} + \{v_m, v_1\}.
\]

Since \( S^1 \) does not have a two-dimensional simplex, \( z \) is a representative element of homology classes of \( S^1 \), \( z \) is transferred by \( (f_0A)_* \) to the following \( z' \).
\[
z' = \{w_1, w_2\} + \{w_2, w_3\} + \{w_3, w_1\}.
\]

9 We can show that individual \( i \) is the dictator in Case 1 when there are only three alternatives by similar procedures.

10 By BA we obtain
\[
x_i^*Rx_k \quad \text{for all} \quad x_j \in A \setminus \{x_1\} \quad \text{or} \quad x_j^*Rx_2 \quad \text{for all} \quad x_j \in A \setminus \{x_2\}.
\]

Then, AC or T (Transitivity) implies (4).
This is a cycle of $S^1$. Therefore, we have

$$(f \circ A)_* \neq 0.$$  \hfill (5)

Now we show the following lemma.

**Lemma 2.** If a BCR satisfies BA, AC, PAR and PR, and has no dictator, then we obtain

$$(f \circ i)_* = 0 \quad \text{for all } i.$$  \hfill (6)

**Proof.** By SPAR when the preference of every individual other than one individual (denoted by $i$) is fixed at $v_1$, the correspondences from the preference of individual $i$ to the social binary relation from $v_1$ to $v_{(m-1)}$ are as follows:

$$v_1 \sim v_{(m-1)} \rightarrow w_1.$$

Lemma 1 implies that the correspondence from $(234 \ldots m1)$ to the social binary relation is as follows:

$$(234 \ldots m1) \rightarrow w_1, \quad \text{and we have } x_1Rx_j, x_jRx_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}.$$  

Then, PR implies that $x_3$ is never the unique best alternative for BCR so long as the most preferred alternative for all individuals other than $i$ is $x_1$ regardless of the preference of individual $i$, and so the preference of individual $i$ corresponds to $w_1$. Thus, we obtain the following correspondences.

$$v_{(m-1)} \sim v_m \rightarrow w_1 \text{ or } w_2.$$

Sets of one-dimensional simplices included in $S^1$ which are one-dimensional cycles are only the following $z$ and its counterpart $-z$.

$$z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m-1}, v_m \rangle + \langle v_m, v_1 \rangle.$$  

Since $S^1$ does not have a two-dimensional simplex, $z$ is a representative element of homology classes of $S^1$. $z$ is transferred by $(f \circ i)_*$ to the following $z'$.

$$z' = \langle w_1, w_1 \rangle = 0 \quad \text{or} \quad z' = \langle w_1, w_2 \rangle + \langle w_2, w_1 \rangle = 0.$$  

Therefore, we have $(f \circ i)_* = 0$ for all $i$. \hfill \Box

The conclusion of this lemma contradicts (2) and (5). Therefore, we have shown Theorem 1. We call the property expressed in (6) the non-surjectivity of individual inclusion mappings. Then, Theorem 1 is a special case of the following theorem.

**Theorem 2.** There exists no binary social choice rule which satisfies SPAR and the non-surjectivity of individual inclusion mappings.

From (5) SPAR implies the surjectivity of the diagonal mapping, $(f \circ A)_* \neq 0$, for binary social choice rules. Thus, this theorem is rewritten as follows: There exists no binary social choice rule which satisfies the surjectivity of the diagonal mapping and the non-surjectivity of individual inclusion mappings.

4. Concluding remarks

In Baryshnikov [3] he said, “the similarities between the two theories, the classical and topological ones, are somewhat more extended than one would expect. The details seem to fit too well to represent just an analogy. I would conjecture that the homological way of proving results in both theories is a ‘true’ one because of its uniformity and thus can lead to much deeper understanding of the structure of social choice. To understand this structure better we need a much more evolved collection of examples of unifying these two theories and I hope this can and will be done.” This paper is an attempt to provide such an example.
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References