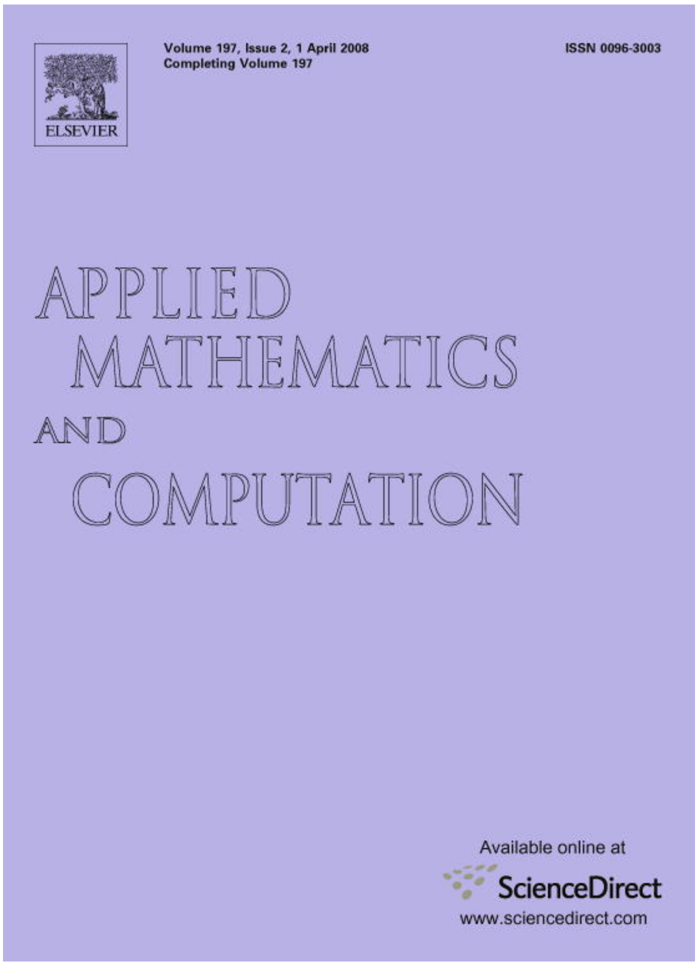
Volume 197, Issue 2, 1 April 2008
Completing Volume 197
ISSN 0096-3003
ELSEVIER
APPLIED MATHEMATICS AND COMPUTATION
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 197 (2008) 598–603
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

On the computability of binary social choice rules in an infinite society and the halting problem ☆

Yasuhito Tanaka

Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto 602-8580, Japan

Abstract

This paper investigates the computability problem of the Arrow impossibility theorem [K.J. Arrow, Social Choice and Individual Values, second ed., Yale University Press, 1963] of social choice theory in a society with an infinite number of individuals (infinite society) according to the computable calculus (or computable analysis) by Aberth [O. Aberth, Computable Analysis, McGraw-Hill, 1980, O. Aberth, Computable Calculus, Academic Press, 2001]. We will show the following results. The problem whether a transitive binary social choice rule satisfying Pareto principle and independence of irrelevant alternatives (IIA) has a dictator or has no dictator in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether the binary social choice rule has a dictator or has no dictator. And it is equivalent to nonsolvability of the halting problem. A binary social choice rule is a function from profiles of individual preferences to social preferences, and a dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter.

Keywords: Arrow impossibility theorem; Infinite society; Nonsolvable problem; Halting problem

1. Introduction

This paper investigates the computability problem of the Arrow impossibility theorem [3] of social choice theory in a society with an infinite number of individuals (infinite society) according to the computable calculus (or computable analysis) by Aberth [1,2]. Arrow's impossibility theorem shows that, with a finite number of individuals, for any binary social choice rule which satisfies the conditions of transitivity, Pareto principle and independence of irrelevant alternatives (IIA) there exists a dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter. On the other hand, [4–6] show that, in a society with an infinite number of individuals (infinite society), there exists a transitive binary social choice rule satisfying Pareto principle and IIA without dictator.[1]

☆ This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid in Japan.
E-mail address: yatanaka@mail.doshisha.ac.jp
[1] Ref. [10] is a recent book that discusses social choice problems in an infinite society.

0096-3003/$ - see front matter © 2007 Published by Elsevier Inc.
doi:10.1016/j.amc.2007.07.059

In the next section, we present the framework of this paper and some preliminary results. In Section 3 we will show the following results. The problem whether a transitive binary social choice rule satisfying Pareto principle and IIA has a dictator or has no dictator in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether the binary social choice rule has a dictator or has no dictator. And it is equivalent to nonsolvability of the halting problem.

2. The framework and preliminary results

There are more than two (finite or infinite) alternatives and a countably infinite number of individuals. The set of individuals is denoted by ω, and the set of alternatives is denoted by A. The alternatives are represented by x, y, z, w and so on. Individual preferences over the alternatives are transitive linear orders, that is, they prefer one alternative to another alternative, and are not indifferent between them. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (\succ_1, \succ_2, \ldots))$, $\mathbf{p}'(= (\succ'_1, \succ'_2, \ldots))$ and so on. We assume that the profiles satisfy the free triple property. It means that about any set of three alternatives, the profiles of individual preferences are not restricted. About a set of three alternative (denoted by $\{x, y, z\}$) we denote the set of preferences of individual i by Σ^i_{xyz}. The set of profiles about $\{x, y, z\}$ is denoted by Σ^ω_{xyz}, where $\omega = \{1, 2, \ldots\}$ is the set of natural numbers. It represents the set of individuals.

We consider a binary social choice rule about $\{x, y, z\}$ $f : \Sigma^\omega_{xyz} \to \Sigma_{xyz}$ which determines a social preference about $\{x, y, z\}$ corresponding to each profile. Σ_{xyz} in this formulation denotes the set of social preferences about $\{x, y, z\}$. We denote $x \succ y$ when the society strictly prefers x to y, and denote $x \sim y$ when the society is indifferent between them. The social preference is denoted by $\succ$ at $\mathbf{p}$, by $\succ'$ at $\mathbf{p}'$ and so on.

The social preferences are required to satisfy *transitivity*, *Pareto principle* and *Independence of irrelevant alternatives (IIA)*. The meanings of these conditions are as follows.

Transitivity. About three alternatives x, y and z, $x \succ y$ and $y \succ z$ (or $x \succ y$ and $y \sim z$, or $x \sim y$ and $y \succ z$) imply $x \succ z$, and $x \sim y$ and $y \sim z$ imply $x \sim z$.

Pareto principle. When all individuals prefer x to y, the society must prefer x to y.

Independence of irrelevant alternatives (*IIA*). The social preference about every pair of two alternatives x and y is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x and y.

Arrow's impossibility theorem shows that, with a finite number of individuals, for any binary social choice rule which satisfies transitivity, Pareto principle and IIA there exists a dictator. In contrast [4–6] show that when the number of individuals in a society is infinite, there exists a transitive binary social choice rule satisfying Pareto principle and IIA without dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter.

Ideal computer. Now we consider an ideal computer according to Aberth [2]. An ideal computer is a machine that manipulates symbol strings, and these symbol strings may be arbitrarily long. The ideal computer may have a finite number of registers. Initially all registers are empty of symbol strings, except for a few registers, $v_1, v_2, \ldots, v_n$, this being the inputs to the ideal computer. The outputs of the ideal computer, after it ceases computation, is the contents of another group of registers, $w_1, w_2, \ldots, w_m$. If P is the program of the ideal computer, with its registers $v_1, v_2, \ldots, v_n$ set to prescribed values $a_1, a_2, \ldots, a_n$, respectively, then $P(a_1, a_2, \ldots, a_n)$ designates its outputs after computation terminates, that is, the values that leave in $w_1, w_2, \ldots, w_m$. An ideal computer for a social choice rule will be explained in the next section.

Next, according to definitions in [8], we define the following terms.

Almost decisiveness. If, when all individuals in a finite or infinite group G prefer an alternative x to another alternative y, and other individuals (individuals in $\omega\backslash G$) prefer y to x, the society prefers x to y ($x \succ y$), then G is *almost decisive* for x against y.
Decisiveness. If, when all individuals in a group G prefer x to y, the society prefers x to y regardless of the preferences of other individuals, then G is *decisive* for x against y.
Decisive set. If a group of individuals is decisive about every pair of alternatives, it is called a decisive set.

A decisive set may consist of one individual. If an individual is decisive about every pair of alternatives for a binary social choice rule, then he is a *dictator* of the binary social choice rule. Of course, there exists at most one dictator.

First about decisiveness we can show the following lemma.

Lemma 1. *If a group of individuals G is almost decisive for an alternative x against another alternative y, then it is decisive about every pair of alternatives, that is, it is a decisive set.*

Proof. See Appendix A. □

The implications of Lemma 1 are similar to those of Lemma 3*a in [8] and Dictator Lemma in [9]. Next we show the following lemma.

Lemma 2. *If G_1 and G_2 are decisive sets, then $G_1 \cap G_2$ is also a decisive set.*

Proof. See Appendix B. □

Note that G_1 and G_2 cannot be disjoint. Assume that G_1 and G_2 are disjoint. If individuals in G_1 prefer x to y, and individuals in G_2 prefer y to x, then neither G_1 nor G_2 can be a decisive set. This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

These are standard results of social choice theory. But for convenience of readers we present the proofs of these lemmas in the appendices.

3. Computability of social choice rules and the halting problem

Consider profiles such that about three alternatives x, y and z one individual (denoted by i) prefers x to y to z, and all other individuals prefer z to x to y. Denote such a profile by $\mathbf{p}^i$, and the set of such profiles is denoted by $\bar{\Sigma}^{\omega}_{x,y,z}$. By Pareto principle, the social preference about x and y is $x \succ y$. The social preference is $x \succ z$ or $z \succ y$.[2] If the social preference is $x \succ z$ at $\mathbf{p}^i$ for some i, then by IIA individual i is almost decisive for x against z, and by Lemma 1, he is a dictator. On the other hand, if the social preference is $z \succ y$ at $\mathbf{p}^i$ for all $i \in N$, then there exists no dictator. In this case by IIA, Lemmas 1 and 2 all co-finite sets (groups of individuals whose complements are finite sets) are decisive sets. Thus, we obtain.

Lemma 3

(1) *Any binary social choice rule which satisfies Pareto principle and IIA has a dictator or has no dictator.*
(2) *In the latter case all co-finite sets are decisive sets.*

We can show, however, that for any transitive binary social choice rule satisfying Pareto principle and IIA, the problem whether it has a dictator or has no dictator is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether it has a dictator or has no dictator.

[2] If $x \sim z$ (or $z \succ x$) and $y \sim z$ (or $y \succ z$), transitivity implies $x \sim y$ (or $y \succ x$). It is a contradiction.

Ideal computer for binary social choice rules. We consider a program P of an ideal computer for such a transitive binary social choice rule restricted to profiles in $\bar{\Sigma}^{\omega}_{x,y,z}$. The input I of P is a string of individual preferences $(\succ_1, \succ_2, \ldots)$. Possible preferences of each individual about x, y and z and also possible social preferences about x, y and z are, respectively, appropriately enumerated. The ideal computer reads the preference of each individual at the profile $\mathbf{p}^i$, $i = 1, 2, \ldots$, step by step from the preference of individual 1 at $\mathbf{p}^1$, and registers them in sequence in the register v_1. It decides the social preference at $\mathbf{p}^i, i = 1, 2, \ldots$, after reading preferences of the first some individuals including individual i, that is, it decides the social preference at $\mathbf{p}^1$ after reading preferences of individuals including individual 1, decides the social preference at $\mathbf{p}^2$ after reading preferences of individuals including individual 2, and so on. And it registers the social preference at each profile in sequence in the register v_2.

If the social preference at $\mathbf{p}^1$ is $x \succ z$, then the ideal computer finds that individual 1 is a dictator, writes "1" in the register w_1 whose value is its output, and it terminates; on the other hand if the social preference at $\mathbf{p}^1$ is $z \succ y$, then the ideal computer does not find a dictator and it continues to read the preference of individual 1 at $\mathbf{p}^2$ in the next step. If the social preference at $\mathbf{p}^2$ is $x \succ z$, then it finds that individual 2 is a dictator, writes "2" in w_1, and it terminates; on the other hand if the social preference at $\mathbf{p}^2$ is $z \succ y$, then it does not find a dictator and it continues to read the preference of individual 1 at $\mathbf{p}^3$ in the next step, and so on. If the binary social choice rule has a dictator, the ideal computer eventually finds a dictator and terminates. On the other hand if the binary social choice rule does not have a dictator, the ideal computer can not find a dictator and it continues computation forever.

We show the following theorem which is the main result of this paper.

Theorem 1

(1) *For any transitive binary social choice rule satisfying Pareto principle and IIA, the problem whether the binary social choice rule has a dictator or has no dictator is a nonsolvable problem, that is, there exists no ideal computer program for any transitive binary social choice rule satisfying Pareto principle and IIA that decides whether it has a dictator or has no dictator.*
(2) *The above result is equivalent to nonsolvability of the halting problem.*

Proof

(1) We assume that there is an ideal computer program P^* which solves the problem whether the ideal computer program P for a transitive binary social choice rule finds a dictator or not, that is, it terminates or not. The inputs to the program P^* are a program P in its register v_1 and a string of individual preferences I, which is the input to P, in v_2. P^* analyzes the program P with the input I, and supplies in w_1 a single output integer having two values, 1 to indicate that P finds a dictator, and 0 to indicate that P does not find a dictator. The 0–1 output of P^* is a function of P and I, and then we denote $P^*(P, I)$.Next we define a program $P'(I)$ such that $P^*(P', I)$ is wrong. First, we construct another program P_S, whose inputs are two programs P^*, P and an integer K. In this formulation, K denotes the maximum number of profiles P has read. Thus, we assume that P reads individual preferences until it decides the social preferences at $\mathbf{p}^i$, $i = 1, 2, \ldots, K$, or P^* terminates before then. The program $P_S(P^*, P(I), K)$ follows the actions of $P^*(P, I)$ step by step. Then, P_S supplies three output integers. The first output integer is 0 if $P^*(P, I)$ does not terminate after P decides the social preference at $\mathbf{p}^K$, and is 1 if $P^*(P, I)$ terminates just when P decides the social preference at $\mathbf{p}^K$ or before then. If the first output integer is 1, the remaining two output integers are significant, one giving the exact number of K, denoted by K^*, taken by $P^*(P, I)$ to termination, and the other giving the $P^*(P, I)$ output integer, 1 or 0, left in w_1 (of P^*). The program $P'(I)$ employs P_S as a subroutine and behaves as follows.
   (i) If P_S signals termination of $P^*(P', I)$ with the output 1 in w_1 (existence of dictator), then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $\mathbf{p}^i, i = 1, 2, \ldots$
   (ii) If P_S signals termination of $P^*(P', I)$ with the output 0 in w_1 (non-existence of dictator), then $P'(I)$ gives the result that the social preference about x and z is $x \succ z$ at $\mathbf{p}^{K^*}$.

(iii) If P_S signals nontermination of $P^*(P', I)$ after P decides the social preference at $\mathbf{p}^K$, then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $\mathbf{p}^i, i = 1, 2, \ldots, K$.

Thus the binary social choice rule has a dictator or has no dictator, depending on whether P^* claims that it has no dictator or has a dictator, respectively. Whatever result P^* determines for P', the program P^* is wrong. And if P^* never terminate, it is still wrong because it fails to give a valid result that the transitive binary social choice rule has no dictator.[3]

(2) According to [2], the halting problem is stated as follows.

Let P be any program that receives its input I in a single register v_1, and P^* be a program with its inputs P in a register v_1 and I in v_2, and supplies in w_1 a single output integer, 1 to indicate termination for P and 0 to indicate nontermination for P. The halting problem is: Is there a program P^* that can determine whether P with that input will terminate or not terminate?

From the arguments before this theorem and the proof of (1) of this theorem it is clear that nonsolvability of the problem whether any transitive binary social choice rule satisfying Pareto principle and IIA has a dictator or has no dictator is equivalent to nonsolvability of the halting problem. □

Note: $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$ for each i. Consider the following profile.

(1) Individual i: $y \succ_i x \succ_i z$.
(2) Other individuals (denoted by j): $z \succ_j y \succ_j x$.

Assume $x \succ z$ and $z \succ y$ at $\mathbf{p}^i$ for some i. By IIA the social preference is $x \succ z$ and $z \succ y$. Then, by transitivity the social preference about x and y must be $x \succ y$. But by Pareto principle, the social preference must be $y \succ x$. Therefore, $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$ for each i.

4. Final remark

We have examined the Arrow impossibility theorem of social choice theory in an infinite society. The assumption of an infinite society seems to be unrealistic. But [7] presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

Appendix A. Proof of Lemma 1

Consider the following profile.

(1) Individuals in G (denoted by i): $x \succ_i y \succ_i z$.
(2) Other individuals (denoted by j): $y \succ_j z$, $y \succ_j x$, and their preferences about x and z are not specified.

By Pareto principle, the social preference is $y \succ z$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by transitivity the social preference should be $x \succ z$. This means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.

(1) Individuals in G (denoted by i): $z \succ_i x \succ_i y$.
(2) Other individuals (denoted by j): $z \succ_j x$, $y \succ_j x$, and their preferences about y and z are not specified.

By similar procedures, we can show that G is decisive for y against z, for z against x, for y against x, and for x against y.

Interchanging z with another alternative $w \neq x, y, z$, we can show that G is decisive about $\{x, y, w\}$. Similarly we can show that G is decisive about $\{x, v, w\}$, is decisive about $\{u, v, w\}$. u, v and w are arbitrary. Therefore, G is decisive about every pair of alternatives. □

---

[3] This proof is based on the proof of nonsolvability of the problem to decide whether any real number equals zero or not in [2].

Appendix B. Proof of Lemma 2

Consider the following profile about x, y and z.

(1) Individuals in $G_1 \backslash G_2$ (denoted by i): $z \succ_i x \succ_i y$.
(2) Individuals in $G_2 \backslash G_1$ (denoted by j): $y \succ_j z \succ_j x$.
(3) Individuals in $G_1 \cap G_2$ (denoted by k): $x \succ_k y \succ_k z$.
(4) Other individuals (denoted by l): $z \succ_l y \succ_l x$.

Since G_1 and G_2 are decisive sets, the social preference is $x \succ y$ and $y \succ z$. Then, by transitivity the social preference about x and z should be $x \succ z$. Only individuals in $G_1 \cap G_2$ prefer x to z, and all other individuals prefer z to x. Thus, $G_1 \cap G_2$ is almost decisive for x against z. Then, by Lemma 1 it is a decisive set. □

References

[1] O. Aberth, Computable Analysis, McGraw-Hill, 1980.
[2] O. Aberth, Computable Calculus, Academic Press, 2001.
[3] K.J. Arrow, Social Choice and Individual Values, second ed., Yale University Press, 1963.
[4] P.C. Fishburn, Arrow's impossibility theorem: concise proof and infinite voters, Journal of Economic Theory 2 (1970) 103–106.
[5] B. Hansson, The existence of group preference functions, Public Choice 28 (1976) 89–98.
[6] A.P. Kirman, D. Sondermann, Arrow's theorem, many agents and invisible dictators, Journal of Economic Theory 5 (1972) 267–277.
[7] R. Mihara, Arrow's theorem and turing computability, Economic Theory 10 (1997) 257–276.
[8] Amartya Sen, Collective Choice and Social Welfare, North-Holland, Amsterdam, 1979.
[9] K. Suzumura, Welfare economics beyond welfarist-consequentialism, Japanese Economic Review 51 (2000) 1–32.
[10] A. Taylor, Social Choice and the Mathematics of Manipulation, Cambridge University Press, 2005.